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Application of Complex Integral Formula on Evaluating Line Integral Problems

Chii-Huei Yu[1]*, Shih-Yin Huang[2]

[1]*Department of Information Technology, Nan Jeon University of Science and Technology, Taiwan
[2]Department of Digital Marketing and Advertising, Nan Jeon University of Science and Technology, Taiwan

ABSTRACT

In this paper, we study two types of two dimensional line integral problems. The closed forms of the two types of line integrals can be determined by using a complex integral formula. In addition, two examples are proposed to do calculation practically. The method adopted in this study is to find solutions through manual calculations and verify the answers using Maple.

Keywords: Line Integral, Complex Integral Formula, Closed Forms, Maple.

I. INTRODUCTION

Calculus and engineering mathematics courses provide many methods to solve the integral problems which include change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. In this paper, we study the following two types of two dimensional line integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_\gamma \left[\begin{array}{l} \frac{1}{2}\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]dx \\ -\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)dy \end{array} \right], \quad (1)$$

and

$$\int_\gamma \left[\begin{array}{l} \tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)dx \\ +\frac{1}{2}\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]dy \end{array} \right], \quad (2)$$

where a,b are real numbers, and $\gamma:[t_1,t_2]\to R^2$ is a piecewise smooth curve in R^2 defined by $\gamma(t)=(x(t),y(t))$ which satisfies $a^2+b^2\neq 0$, $[x(t)]^2+[y(t)]^2\neq a^2+b^2$, $[x(t)]^2-[y(t)]^2+a^2-b^2>0$. The two types of line integrals can be determined by using a complex integral formula; these are the main results of this paper (i.e., Theorems 1 and 2). Adams et al. [1], Nyblom [2], and Oster [3] provided some methods to solve the integral problems. Moreover, Yu [4-30], Yu and Chen [31], and Yu and Sheu [32-34] used some techniques to solve some types of integrals, which including complex power series, integration term by term theorem, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula. In this study, we propose some examples to demonstrate the manual calculations, and verify the results using Maple.

II. METHODS AND MATERIAL

First, we introduce two formulas used in this paper.

Formulas:

Suppose that z is a complex number, then

1) $\tan^{-1}z=\frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right)$, for $z\neq -i$. (3)

And

2) $\tanh^{-1}z=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$, for $z\neq 1$. (4)

To obtain the major results, two lemmas are needed and the first one is the complex integral formula used in this article.

Lemma 1 *Suppose that* z, λ *are complex numbers with* $\lambda \neq 0$, *then*

$$\int \ln(z^2+\lambda^2)dz$$

$$= z\ln(z^2+\lambda^2) + 2\lambda\tan^{-1}\frac{z}{\lambda} - 2z + C, \quad (5)$$

where C *is a constant.*

Proof $\int \ln(z^2+\lambda^2)dz$

$$= z\ln(z^2+\lambda^2) - \int z\cdot\frac{2z}{z^2+\lambda^2}dz$$

$$= z\ln(z^2+\lambda^2) - \int\left(2-\frac{2\lambda^2}{z^2+\lambda^2}\right)dz$$

$$= z\ln(z^2+\lambda^2) + 2\lambda^2\int\frac{1}{z^2+\lambda^2}dz - 2z$$

$$= z\ln(z^2+\lambda^2) + 2\lambda\tan^{-1}\frac{z}{\lambda} - 2z + C.$$

q.e.d.

Lemma 2 *Suppose that* α, β *are real numbers with* $\alpha^2+\beta^2 \neq 1$, *then*

$$\tan^{-1}(\alpha+i\beta)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right) + i\frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right). \quad (6)$$

Proof Using Eq. (3) yields

$$\tan^{-1}(\alpha+i\beta)$$

$$= \frac{1}{2i}\ln\left(\frac{1-\beta+i\alpha}{1+\beta-i\alpha}\right)$$

$$= \frac{1}{2i}\ln\left(\frac{(1-\alpha^2-\beta^2)+i2\alpha}{(1+\beta)^2+\alpha^2}\right)$$

$$= \frac{1}{2i}\ln\left[\frac{\sqrt{(1-\alpha^2-\beta^2)^2+4\alpha^2}}{(1+\beta)^2+\alpha^2}\times\left(\frac{1-\alpha^2-\beta^2}{\sqrt{(1-\alpha^2-\beta^2)^2+4\alpha^2}} + i\frac{2\alpha}{\sqrt{(1-\alpha^2-\beta^2)^2+4\alpha^2}}\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right) - i\frac{1}{2}\ln\left(\frac{\sqrt{(1-\alpha^2-\beta^2)^2+4\alpha^2}}{(1+\beta)^2+\alpha^2}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right) + i\frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right).$$

(by Eq. (4)) q.e.d.

III. RESULTS AND DISCUSSION

Main Results

In the following, we use Lemmas 1 and 2 to obtain the closed forms of the line integrals (1) and (2).

Theorem 1 *If* a, b *are real numbers,* $a^2+b^2 \neq 0$ C *is a constant, and let* $\gamma:[t_1,t_2]\to R^2$ *be a piecewise smooth curve in* R^2 *defined by* $\gamma(t)=(x(t),y(t))$ *for* $t\in[t_1,t_2]$, *which satisfies* $[x(t)]^2+[y(t)]^2 \neq a^2+b^2$, *and* $[x(t)]^2-[y(t)]^2+a^2-b^2>0$, *then*

$$\int_\gamma\left[\begin{array}{l}\frac{1}{2}\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]dx \\ -\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)dy\end{array}\right]$$

$$= F(x(t_2),y(t_2)) - F(x(t_1),y(t_1)), \quad (7)$$

where $F(x,y)$

$$= \frac{1}{2}x\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]$$

$$- y\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)$$

$$+ a\tan^{-1}\left(\frac{2(ax+by)}{(a^2+b^2)-(x^2+y^2)}\right)$$

$$- b\tanh^{-1}\left(\frac{2(-bx+ay)}{(a^2+b^2)+(x^2+y^2)}\right) - 2x + C.$$

Proof Let $z = x+iy$ and $\lambda = a+ib$ in Eq. (5), then

$$\int \ln[(x+iy)^2+(a+ib)^2]d(x+iy)$$

$$= (x+iy)\ln[(x+iy)^2+(a+ib)^2]$$
$$+ 2(a+ib)\tan^{-1}\left(\frac{x+iy}{a+ib}\right) - 2(x+iy) + C.$$

Therefore,

$$\int \ln[(x^2-y^2+a^2-b^2)+i(2xy+2ab)](dx+idy)$$

$$= (x+iy)\ln[(x^2-y^2+a^2-b^2)+i(2xy+2ab)]$$
$$+2(a+ib)\tan^{-1}\left(\frac{(ax+by)+i(-bx+ay)}{a^2+b^2}\right)-(2x+i2y)+C. \quad (8)$$

By the equality of the real parts of both sides of Eq. (8), we obtain

$$\int \left[\begin{array}{l}\frac{1}{2}\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]dx \\ -\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)dy\end{array}\right]$$

$$=\frac{1}{2}x\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]$$
$$-y\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)$$
$$+a\tan^{-1}\left(\frac{2(ax+by)}{(a^2+b^2)-(x^2+y^2)}\right)$$
$$-b\tanh^{-1}\left(\frac{2(-bx+ay)}{(a^2+b^2)+(x^2+y^2)}\right)-2x+C.$$

Therefore, the desired result holds. q.e.d.

Theorem 2 *If the assumptions are the same as Theorem 1 , then*

$$\int_\gamma \left[\begin{array}{l}\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)dx \\ +\frac{1}{2}\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]dy\end{array}\right]$$
$$=G(x(t_2),y(t_2))-G(x(t_1),y(t_1)), \quad (9)$$

where $G(x,y)$

$$=\frac{1}{2}y\ln[(x^2-y^2+a^2-b^2)^2+(2xy+2ab)^2]$$
$$+x\tan^{-1}\left(\frac{2xy+2ab}{x^2-y^2+a^2-b^2}\right)$$
$$+b\tan^{-1}\left(\frac{2(ax+by)}{(a^2+b^2)-(x^2+y^2)}\right)$$
$$+a\tanh^{-1}\left(\frac{2(-bx+ay)}{(a^2+b^2)+(x^2+y^2)}\right)-2y+C.$$

Proof By the equality of the imaginary parts of both sides of Eq. (8), the desired result holds. q.e.d.

Examples

For the line integral problems discussed in this study, two examples are proposed and Theorems 1 and 2 are used to determine their closed forms. On the other hand, Maple is used to calculate the approximations of some line integrals and their closed forms to verify our answers.

Example 1 If $a=1, b=1$ in Theorem 1, and let $\gamma:[1,2]\to R^2$ be a piecewise smooth curve defined by $\gamma(t)=(2t,t)$, then using Theorem 1 yields

$$\int_\gamma \left[\begin{array}{l}\frac{1}{2}\ln[(x^2-y^2)^2+(2xy+2)^2]dx \\ -\tan^{-1}\left(\frac{2xy+2}{x^2-y^2}\right)dy\end{array}\right]$$
$$=F(4,2)-F(2,1), \quad (10)$$

where $F(x,y)$

$$=\frac{1}{2}x\ln[(x^2-y^2)^2+(2xy+2)^2]$$
$$-y\tan^{-1}\left(\frac{2xy+2}{x^2-y^2}\right)+\tan^{-1}\left(\frac{2(x+y)}{2-(x^2+y^2)}\right)$$
$$-\tanh^{-1}\left(\frac{2(-x+y)}{2+(x^2+y^2)}\right)-2x+C.$$

That is,

$$\int_1^2\left[\ln(25t^4+16t^2+4)-\tan^{-1}\left(\frac{4t^2+2}{3t^2}\right)\right]dt$$
$$=2\ln 468-2\tan^{-1}\left(\frac{3}{2}\right)+\tan^{-1}\left(\frac{2}{-3}\right)$$
$$-\tanh^{-1}\left(\frac{-2}{11}\right)-\ln 45+2\tan^{-1}2$$
$$+\tanh^{-1}\left(\frac{-2}{7}\right)-4. \quad (11)$$

Using Maple to verify the correctness of Eq. (11) as follows:

```
>evalf(int(ln(25*t^4+16*t^2+4)-arctan((4*t^2+2)/(3*t^2)),t=1..2),18);
```

4.04095054522252823

```
>evalf(2*ln(468)-2*arctan(3/2)+arctan(-2/3)-arctanh(-2/11)-ln(45)+2*arctan(2)+arctanh(-2/7)-4,18);
```

4.04095054522252827

Example 2 In Theorem 2, if $a=3, b=2$, and let $\gamma:[1,3]\to R^2$ be a piecewise smooth curve defined by $\gamma(t)=(4t,t)$, then by Theorem 2 we have

$$\int_\gamma \left[\tan^{-1}\left(\frac{2xy+12}{x^2-y^2+5}\right)dx + \frac{1}{2}\ln[(x^2-y^2+5)^2+(2xy+12)^2]dy \right]$$

$$=G(12,3)-G(4,1), \qquad (12)$$

where $G(x,y)$

$$=\frac{1}{2}y\ln[(x^2-y^2+5)^2+(2xy+12)^2]$$

$$+x\tan^{-1}\left(\frac{2xy+12}{x^2-y^2+5}\right)$$

$$+2\tan^{-1}\left(\frac{2(3x+2y)}{13-(x^2+y^2)}\right)$$

$$+3\tanh^{-1}\left(\frac{2(-2x+3y)}{13+(x^2+y^2)}\right)-2y+C.$$

Therefore,

$$\int_1^3\left[4\tan^{-1}\left(\frac{8t^2+12}{15t^2+5}\right)+\frac{1}{2}\ln(289t^4+342t^2+169)\right]dt$$

$$=\frac{3}{2}\ln 26656+10\tan^{-1}\left(\frac{3}{5}\right)+3\tanh^{-1}\left(\frac{-15}{83}\right)-4$$

$$-\frac{1}{2}\ln 800-\pi+2\tan^{-1}7-3\tanh^{-1}\left(\frac{-1}{3}\right). \qquad (13)$$

We also employ Maple to verify the correctness of Eq. (13).

```
>evalf(int(4*arctan((8*t^2+12)/(15*t^2+5))+1/2*ln(289*t^4+342*t^2+169),t=1..3),18);
```

13.5557804577116240

```
>evalf(3/2*ln(26656)+10*arctan(3/5)+3*arctanh(-15/83)-4-1/2*ln(800)-Pi+2*arctan(7)-3*arctanh(-1/3),18);
```

13.5557804577116242

IV.CONCLUSION

As mentioned, we mainly use a complex integral formula to solve two types of two dimensional line integrals. In fact, the applications of complex integral formulas are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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