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A Study of Definite Integrals Using Parseval's Identity

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ABSTRACT

This paper studies two types of definite integrals. Using Parseval's identity, we can determine the infinite series expressions of the two types of definite integrals. Moreover, we propose two examples to do calculation practically. The research method adopted in this study is to find solutions through manual calculations and verify these solutions using Maple. This research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords: Definite Integrals, Parseval's Identity, Infinite Series Expressions, Maple.

I. INTRODUCTION

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics courses, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. Adams et al. [1], Nyblom [2], and Oster [3] provided some methods to solve the integral problems. On the other hand, Yu [4-34], Yu and Chen [35], and Yu and Sheu [36-38] used some techniques, for example, complex power series, integration term by term theorem, area mean value theorem, and generalized Cauchy integral formula to solve some types of integrals. In this article, we study the following two types of definite integral problems which are not easy to obtain its answer using the methods mentioned above.

$$\int_0^{2\pi} \exp[s\cos(x+\varphi)] \begin{bmatrix} \frac{1}{2}\cos[s\sin(x+\varphi)] \\ \times \ln(1+2r\cos x + r^2) \\ +\sin[s\sin(x+\varphi)] \\ \times \tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right) \end{bmatrix} dx, \tag{1}$$

$$\int_0^{2\pi} \exp[s\cos(x+\varphi)] \begin{bmatrix} -\frac{1}{2}\sin[s\sin(x+\varphi)] \\ \times \ln(1+2r\cos x + r^2) \\ +\cos[s\sin(x+\varphi)] \\ \times \tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right) \end{bmatrix} dx, \tag{2}$$

where r, s, φ are real number, and $|r|<1$. The infinite series expressions of the two types of definite integrals can be obtained by using Parseval's identity; they are the major results of this paper (i.e., Theorems 1 and 2). In addition, we propose two examples to demonstrate the manual calculations, and verify the results using Maple.

II. METHODS AND MATERIAL

First, some formulas and Parseval's identity used in this paper are introduced below.

Formulas

1) Euler's formula

$\exp(ix)=\cos x+i\sin x$, where $i=\sqrt{-1}$ and x is a real number.

2) DeMoivre's formula

$(\cos x+i\sin x)^p=\cos px+i\sin px$, where p is an integer, and x is a real number.

3) Taylor series expansions

$$\ln(1+z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}, \tag{3}$$

where z is a complex number, and $|z|<1$.

And

$$\exp(\lambda z)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}z^n, \tag{4}$$

where λ, z are complex numbers.

Parseval's identity

If $f(x)$ and $g(x)$ are two square integrable (with respect to Lebesgue measure), complex valued functions defined on R of period 2π with Fourier series expansions

$$f(x)=\sum_{n=-\infty}^{\infty}a_n\exp(inx),$$

$$g(x)=\sum_{n=-\infty}^{\infty}b_n\exp(inx),$$

respectively, then

$$\frac{1}{2\pi}\int_0^{2\pi}f(x)\overline{g(x)}dx=\sum_{n=-\infty}^{\infty}a_n\overline{b_n}. \tag{5}$$

III. RESULTS AND DISCUSSION

Main Results

In the following, we use Parseval's identity to determine the infinite series expressions of the definite integrals (1) and (2).

Theorem 1 *Suppose that r, s, φ are real numbers, and $|r|<1$, then*

$$\int_0^{2\pi}\exp[s\cos(x+\varphi)]\begin{bmatrix}\frac{1}{2}\cos[s\sin(x+\varphi)]\\ \times\ln(1+2r\cos x+r^2)\\ +\sin[s\sin(x+\varphi)]\\ \times\tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right)\end{bmatrix}dx$$

$$=-2\pi\sum_{n=1}^{\infty}\frac{(-rs)^n\cos n\varphi}{n\cdot n!}. \tag{6}$$

Proof Let $\lambda=\exp(i\varphi)$, $f(x)=\ln[1+r\exp(ix)]$ and $g(x)=\exp[\lambda s\exp(ix)]$, then

$$f(x)\overline{g(x)}$$

$$=\ln[1+r\exp(ix)]\cdot\exp\{s\exp[-i(x+\varphi)]\}$$

$$=\ln[(1+r\cos x)+ir\sin x]$$
$$\times\exp[s\cos(x+\varphi)-is\sin(x+\varphi)]$$

(by Euler's formula)

$$=\left[\frac{1}{2}\ln[(1+r\cos x)^2+(r\sin x)^2]+i\tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right)\right]$$
$$\times\exp[s\cos(x+\varphi)]$$
$$\times\{\cos[s\sin(x+\varphi)]-i\sin[s\sin(x+\varphi)]\}$$

$$=\exp[s\cos(x+\varphi)]\begin{bmatrix}\frac{1}{2}\cos[s\sin(x+\varphi)]\\ \times\ln(1+2r\cos x+r^2)\\ +\sin[s\sin(x+\varphi)]\\ \times\tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right)\end{bmatrix}$$
$$+i\exp[s\cos(x+\varphi)]\begin{bmatrix}-\frac{1}{2}\sin[s\sin(x+\varphi)]\\ \times\ln(1+2r\cos x+r^2)\\ +\cos[s\sin(x+\varphi)]\\ \times\tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right)\end{bmatrix}. \tag{7}$$

On the other hand, by Eqs. (3), (4) and DeMoivre's formula, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}[r\exp(ix)]^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n r^{n+1}}{n+1}\exp[i(n+1)x]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} r^n}{n}\exp(inx),$$

and

$$g(x)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}[\lambda s\exp(ix)]^n$$

$$= \sum_{n=0}^{\infty} \frac{s^n \exp(in\varphi)}{n!}\exp(inx)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{s^n \exp(in\varphi)}{n!}\exp(inx)$$

Thus, by Parseval's identity we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$$

$$= -\sum_{n=1}^{\infty} \frac{(-rs)^n \exp(-in\varphi)}{n \cdot n!}. \qquad (8)$$

It follows from Eq. (7) and the real parts of both sides of Eq. (8) are equal that the desired result holds. q.e.d.

Theorem 2 *If the assumptions are the same as Theorem 1, then*

$$\int_0^{2\pi} \exp[s\cos(x+\varphi)] \begin{bmatrix} -\frac{1}{2}\sin[s\sin(x+\varphi)] \\ \times \ln(1+2r\cos x + r^2) \\ +\cos[s\sin(x+\varphi)] \\ \times \tan^{-1}\left(\frac{r\sin x}{1+r\cos x}\right) \end{bmatrix} dx$$

$$= 2\pi \sum_{n=1}^{\infty} \frac{(-rs)^n \sin n\varphi}{n \cdot n!}. \qquad (9)$$

Proof Using Eq. (7) and by the imaginary parts of both sides of Eq. (8) are equal, the desired result holds.

q.e.d.

Examples

For the definite integral problems discussed in this study, two examples are provided and we use Theorems 1 and 2 to obtain their infinite series expressions. Moreover, Maple is used to calculate the approximations of these definite integrals and their infinite series expressions for verifying our answers.

Example 1 Let $r = 1/4$, $s = 5$, and $\varphi = \pi/3$ in Theorem 1, then by Eq. (6) we have

$$\int_0^{2\pi} \exp[5\cos(x+\pi/3)] \begin{bmatrix} \frac{1}{2}\cos[5\sin(x+\pi/3)] \\ \times \ln(17/16 + 1/2\cos x) \\ +\sin[5\sin(x+\pi/3)] \\ \times \tan^{-1}\left(\frac{\sin x}{4+\cos x}\right) \end{bmatrix} dx$$

$$= -2\pi \sum_{n=1}^{\infty} \frac{(-5/4)^n \cos(n\pi/3)}{n \cdot n!}. \qquad (10)$$

Next, we employ Maple to verify the correctness of Eq. (10).

>evalf(int(exp(5*cos(x+Pi/3))*(1/2*cos(5*sin(x+Pi/3))*ln(17/16+1/2*cos(x))+sin(5*sin(x+Pi/3))*arctan(sin(x)/(4+cos(x)))),x=0..2*Pi),16);

4.563200959177383

>evalf(-2*Pi*sum((-5/4)^n*cos(n*Pi/3)/(n*n!),n=1..infinity),16);

4.563200959177382

Example 2 In Theorem 2, if $r = 1/3$, $s = 7$, and $\varphi = \pi/4$, then using Eq. (9) yields

$$\int_0^{2\pi} \exp[7\cos(x+\pi/4)] \begin{bmatrix} -\frac{1}{2}\sin[7\sin(x+\pi/4)] \\ \times \ln(10/9 + 2/3\cos x) \\ +\cos[7\sin(x+\pi/4)] \\ \times \tan^{-1}\left(\frac{\sin x}{3+\cos x}\right) \end{bmatrix} dx$$

$$= 2\pi \sum_{n=1}^{\infty} \frac{(-7/3)^n \sin(n\pi/4)}{n \cdot n!}. \qquad (11)$$

We also use Maple to verify the correctness of Eq. (11).

>evalf(int(exp(7*cos(x+Pi/4))*(-1/2*sin(7*sin(x+Pi/4))*ln(10/9+2/3*cos(x))+cos(7*sin(x+Pi/4))*arctan(sin(x)/(3+cos(x)))),x=0..2*Pi),16);

-4.627451284683444

>evalf(2*Pi*sum((-7/3)^n*sin(n*Pi/4)/(n*n!),n=1..infinity),16);

-4.627451284683444

IV.CONCLUSION

In this article, we mainly use Parseval's identity to solve two types of definite integrals. In fact, the applications of Parseval's identity are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. Moreover, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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