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# Calculation of Partial Derivatives of Two Variables Functions

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## ABSTRACT

In this article, we study the partial differential problem of two types of two variables functions. The infinite series forms of any order partial derivatives of the two types of two variables functions can be obtained mainly using differentiation term by term theorem. Therefore, the difficulty of calculating higher order partial derivatives of these two variables functions can be greatly reduced. On the other hand, some examples are provided to do calculation practically. The research method adopted is to find solutions through manual calculations, and verify these solutions using Maple.

**Keywords :** Two Variables Functions, Partial Derivatives, Infinite Series Forms, Differentiation Term By Term Theorem, Maple.

## I. INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In mathematics and physics, the study of partial differential problem of multivariable functions is very important. For example, Laplace equations, wave equations, and other physical equations are involved the partial derivatives of multivariable functions. Therefore, in many scientific fields, the evaluation and numerical calculations of partial derivatives of multivariable functions are necessary. This paper studies the partial differential problem of the following two types of two variables functions:

$$f(x,y) = (\alpha^2x^2+\beta^2y^2)^{p/2}\times\exp\left(\frac{(\alpha r\cos\varphi)x+(\beta r\sin\varphi)y}{\alpha^2x^2+\beta^2y^2}\right)\times\cos\left(\frac{(\alpha r\sin\varphi)x-(\beta r\cos\varphi)y}{\alpha^2x^2+\beta^2y^2}+p\tan^{-1}\frac{\beta y}{\alpha x}\right) \quad (1)$$

and

$$g(x,y) = (\alpha^2x^2+\beta^2y^2)^{p/2}\times\exp\left(\frac{(\alpha r\cos\varphi)x+(\beta r\sin\varphi)y}{\alpha^2x^2+\beta^2y^2}\right)\times\sin\left(\frac{(\alpha r\sin\varphi)x-(\beta r\cos\varphi)y}{\alpha^2x^2+\beta^2y^2}+p\tan^{-1}\frac{\beta y}{\alpha x}\right), \quad (2)$$

where $\alpha, \beta, r, \varphi, x, y$ are real numbers, $\alpha \neq 0$, and $\alpha x > 0$. Using differentiation term by term theorem, we can determine the infinite series forms of any order partial derivatives of the two types of two variables functions. Thus, we can greatly reduce the difficulty of evaluating the higher order partial derivatives of these two variables functions. [1-5] provided some methods to evaluate the partial derivatives of multivariable functions, which are different from the methods used in this paper. In addition, [6-23] used some techniques, for example, complex power series, binomial series, and differentiation term by term theorem to study the partial differential problem. In this study, we propose two examples of two variables functions to evaluate their any order partial derivatives, and calculate some of their higher order partial derivative values practically. Moreover, we employ Maple to calculate the approximations of these higher order partial derivative values and their infinite series forms to verify our answers.

## II. METHODS AND MATERIAL

**First, some properties used in this paper are introduced below.**

**Definitions**

**1)** Let $m, n$ be non-negative integers. The $(m+n)$-th order partial derivative ($n$-times partial derivatives with respect to $x$, $m$-times partial derivatives with respect to $y$) of a two variables function $f(x, y)$, is denoted by $\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x, y)$.

**2)** Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$ and $a, b$ are real numbers. $a$, the real part of $z$, is denoted as $\operatorname{Re}(z)$; $b$, the imaginary part of $z$, is denoted as $\operatorname{Im}(z)$.

**3)** Assume that $r$ is a real number and $k$ is a positive integer. Define $(r)_k = r(r-1)\cdots(r-k+1)$, and $(r)_0 = 1$.

**Formulas**

**1)Euler's formula**

$e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta$ is any real number.

**2) DeMoivre's formula**

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where $n$ is any integer, and $\theta$ is any real number.

**3)Differentiation term by term theorem** ([24, p230])

If, for all non-negative integer $k$, the functions $g_k : (a, b) \to R$ satisfy the following three conditions：(i) there exists a point $x_0 \in (a, b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval $(a, b)$, (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on $(a, b)$. Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on $(a, b)$. Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

## III. RESULTS AND DISCUSSION

**Main Results**

In the following, we determine the infinite series forms of any order partial derivatives of the two variables functions (1) and (2) respectively.

**Theorem 1** *If $\alpha, \beta, r, \varphi$ are real numbers, $\alpha \neq 0$, $p$ is an integer, $m, n$ are non-negative integers, and let the domain of*

$$f(x, y) = (\alpha^2 x^2 + \beta^2 y^2)^{p/2} \times \exp\left(\frac{(\alpha r\cos\varphi)x + (\beta r\sin\varphi)y}{\alpha^2 x^2 + \beta^2 y^2}\right) \times \cos\left(\frac{(\alpha r\sin\varphi)x - (\beta r\cos\varphi)y}{\alpha^2 x^2 + \beta^2 y^2} + p\tan^{-1}\frac{\beta y}{\alpha x}\right)$$

*be $\left\{(x, y) \in R^2 \middle| \alpha x > 0\right\}$, then*

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x, y) = \sum_{k=0}^{\infty}\left[\begin{array}{l}\frac{\alpha^n \beta^m r^k (p-k)_n (p-k-n)_m}{k!} \\ \times(\alpha^2 x^2 + \beta^2 y^2)^{(p-k-m-n)/2} \\ \times\cos\left((p-k-m-n)\tan^{-1}\frac{\beta y}{\alpha x} + k\varphi + \frac{m\pi}{2}\right)\end{array}\right], \quad (3)$$

*for all* $\alpha x > 0$.

**Proof** Since

$$z^p \exp\left(\frac{\lambda}{z}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} z^{p-k}, \qquad (4)$$

for all complex numbers $z, \lambda$, $z \neq 0$, and any integer $p$.

Let $z = \alpha x + i\beta y$ and $\lambda = re^{i\varphi}$ in Eq. (4), then

$$(\alpha x + i\beta y)^p \exp\left(\frac{re^{i\varphi}}{\alpha x + i\beta y}\right)$$
$$= \sum_{k=0}^{\infty} \frac{(re^{i\varphi})^k}{k!} (\alpha x + i\beta y)^{p-k}.$$

Using Euler's formula and DeMoivre's formula yields

$$(\alpha^2 x^2 + \beta^2 y^2)^{p/2} \exp\left(ip \tan^{-1}\frac{\beta y}{\alpha x}\right)$$
$$\times \exp\left(\frac{(r\cos\varphi + ir\sin\varphi)(\alpha x - i\beta y)}{\alpha^2 x^2 + \beta^2 y^2}\right)$$
$$= \sum_{k=0}^{\infty} \frac{r^k e^{ik\varphi}}{k!} (\alpha x + i\beta y)^{p-k}.$$

Thus,

$$(\alpha^2 x^2 + \beta^2 y^2)^{p/2} \times \exp\left(\frac{(\alpha r\cos\varphi)x + (\beta r\sin\varphi)y}{\alpha^2 x^2 + \beta^2 y^2}\right)$$
$$\times \exp i\left(\frac{(\alpha r\sin\varphi)x - (\beta r\cos\varphi)y}{\alpha^2 x^2 + \beta^2 y^2} + p\tan^{-1}\frac{\beta y}{\alpha x}\right)$$
$$= \sum_{k=0}^{\infty} \frac{r^k e^{ik\varphi}}{k!} (\alpha x + i\beta y)^{p-k}. \qquad (5)$$

And hence, using the equality of the real parts of both sides of Eq. (5) yields

$$f(x,y) = \mathrm{Re}\left[\sum_{k=0}^{\infty} \frac{r^k e^{ik\varphi}}{k!} (\alpha x + i\beta y)^{p-k}\right]. \qquad (6)$$

Differentiating $n$ -times with respect to $x$, and $m$ -times with respect to $y$ in Eq.(6), and using differentiation term by term theorem yields

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y)$$
$$= \mathrm{Re}\left[\sum_{k=0}^{\infty} \frac{\alpha^n (i\beta)^m r^k e^{ik\varphi} (p-k)_n (p-k-n)_m}{k!} \times (\alpha x + i\beta y)^{p-k-m-n}\right]$$
$$= \sum_{k=0}^{\infty} \mathrm{Re}\left[\begin{array}{l} \dfrac{\alpha^n \beta^m r^k (p-k)_n (p-k-n)_m}{k!} \\ \times (\alpha^2 x^2 + \beta^2 y^2)^{(p-k-m-n)/2} \\ \times \exp i\left((p-k-m-n)\tan^{-1}\dfrac{\beta y}{\alpha x} + k\varphi + \dfrac{m\pi}{2}\right) \end{array}\right]$$
$$= \sum_{k=0}^{\infty} \left[\begin{array}{l} \dfrac{\alpha^n \beta^m r^k (p-k)_n (p-k-n)_m}{k!} \\ \times (\alpha^2 x^2 + \beta^2 y^2)^{(p-k-m-n)/2} \\ \times \cos\left((p-k-m-n)\tan^{-1}\dfrac{\beta y}{\alpha x} + k\varphi + \dfrac{m\pi}{2}\right) \end{array}\right].$$

q.e.d.

**Theorem 2** *If the assumptions are the same as Theorem 1 and let the domain of*

$$g(x,y)$$
$$= (\alpha^2 x^2 + \beta^2 y^2)^{p/2} \times \exp\left(\frac{(\alpha r\cos\varphi)x + (\beta r\sin\varphi)y}{\alpha^2 x^2 + \beta^2 y^2}\right)$$
$$\times \sin\left(\frac{(\alpha r\sin\varphi)x - (\beta r\cos\varphi)y}{\alpha^2 x^2 + \beta^2 y^2} + p\tan^{-1}\frac{\beta y}{\alpha x}\right)$$

*be* $\left\{(x,y) \in R^2 \middle| \alpha x > 0\right\}$, *then*

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y)$$
$$= \sum_{k=0}^{\infty} \left[\begin{array}{l} \dfrac{\alpha^n \beta^m r^k (p-k)_n (p-k-n)_m}{k!} \\ \times (\alpha^2 x^2 + \beta^2 y^2)^{(p-k-m-n)/2} \\ \times \sin\left((p-k-m-n)\tan^{-1}\dfrac{\beta y}{\alpha x} + k\varphi + \dfrac{m\pi}{2}\right) \end{array}\right], \qquad (7)$$

*for all* $\alpha x > 0$.

**Proof** By the equality of the imaginary parts of both sides of Eq. (5), we obtain

$$g(x,y) = \mathrm{Im}\left[\sum_{k=0}^{\infty} \frac{r^k e^{ik\varphi}}{k!} (\alpha x + i\beta y)^{p-k}\right]. \qquad (8)$$

Also, using differentiation term by term theorem in Eq. (8) yields the desired result holds. q.e.d.

**Examples**

For the partial differential problem discussed in this article, two functions are proposed and we use Theorems 1 and 2 to obtain the infinite series forms of their any order partial derivatives. On the other hand, Maple is used to calculate the approximations of some partial derivative values and their infinite series forms for verifying our answers.

**Example 1** Suppose that the domain of the two variables function

$$f(x,y) = (16x^2+9y^2)^{5/2}\times\exp\left(\frac{4x+3\sqrt{3}y}{16x^2+9y^2}\right)\times\cos\left(\frac{4\sqrt{3}x-3y}{16x^2+9y^2}+5\tan^{-1}\frac{3y}{4x}\right) \quad (9)$$

is $\left\{(x,y)\in R^2 \middle| x>0\right\}$ (for $\alpha=4,\beta=3,r=2,\varphi=\pi/3$ and $p=5$ in Theorem 1), then by Eq. (3) we have

$$\frac{\partial^{m+n}f}{\partial y^m\partial x^n}(x,y) = \sum_{k=0}^{\infty}\left[\begin{array}{l}\frac{4^n3^m2^k(5-k)_n(5-k-n)_m}{k!}\\ \times(16x^2+9y^2)^{(5-k-m-n)/2}\\ \times\cos\left((5-k-m-n)\tan^{-1}\frac{3y}{4x}+\frac{k\pi}{3}+\frac{m\pi}{2}\right)\end{array}\right], \quad (10)$$

for all $x>0$.

Hence, the 9-th order partial derivative value of $f(x,y)$ at $(5,3)$

$$\frac{\partial^9 f}{\partial y^3\partial x^6}(5,3) = \sum_{k=0}^{\infty}\left[\begin{array}{l}\frac{4^63^32^k(5-k)_6(-1-k)_3}{k!}\\ \times 481^{(-4-k)/2}\\ \times\sin\left((-4-k)\tan^{-1}\frac{9}{20}+\frac{k\pi}{3}\right)\end{array}\right]. \quad (11)$$

Next, we use Maple to verify the correctness of Eq. (11).

```
>f:=(x,y)->(16*x^2+9*y^2)^(5/2)*exp((4*x+3*sqrt(3)*y)/(16*x^2+9*y^2))*cos((4*sqrt(3)*x-3*y)/(16*x^2+9*y^2)+5*arctan(3*y/(4*x)));

>evalf(D[1$6,2$3](f)(5,3),18);

-0.130485701544445346789 62

>evalf(sum(4^6*3^3*2^k*product(5-k-j,j=0..5)*product(-1-k-t,t=0..2)/k!*481^((-4-k)/2)*sin((-4-k)*arctan(9/20)+k*Pi/3),k=0..infinity),18);

-0.130485701544445346789 05
```

**Example 2** If the domain of the two variables function

$$g(x,y) = (4x^2+25y^2)^{7/2}\times\exp\left(\frac{15y}{4x^2+25y^2}\right)\times\sin\left(\frac{-6x}{4x^2+25y^2}+7\tan^{-1}\left(\frac{5y}{-2x}\right)\right) \quad (12)$$

is $\left\{(x,y)\in R^2 \middle| x<0\right\}$ (for $\alpha=-2,\beta=5,r=3,\varphi=\pi/2$, and $p=7$ in Theorem 2), then using Eq. (7) yields

$$\frac{\partial^{m+n}g}{\partial y^m\partial x^n}(x,y) = \sum_{k=0}^{\infty}\left[\begin{array}{l}\frac{(-2)^n5^m3^k(7-k)_n(7-k-n)_m}{k!}\\ \times(4x^2+25y^2)^{(7-k-m-n)/2}\\ \times\sin\left((7-k-m-n)\tan^{-1}\left(\frac{5y}{-2x}\right)+k\varphi+\frac{m\pi}{2}\right)\end{array}\right], \quad (13)$$

for all $x<0$.

Therefore, the 11-th order partial derivative value of $g(x,y)$ at $(-2,4)$

$$\frac{\partial^{11}g}{\partial y^4\partial x^7}(-2,4) = \sum_{k=0}^{\infty}\left[\begin{array}{l}\frac{(-2)^75^43^k(7-k)_7(-k)_4}{k!}\\ \times 416^{(-4-k)/2}\\ \times\sin\left((-4-k)\tan^{-1}5+\frac{k\pi}{2}\right)\end{array}\right]. \quad (14)$$

We also employ Maple to verify the correctness of Eq. (14).

```
>g:=(x,y)->(4*x^2+25*y^2)^(7/2)*exp(15*y/(4*x^2+25*y^2))*sin(-6*x/(4*x^2+25*y^2)+7*arctan(5*y/(-2*x)));

>evalf(D[1$7,2$4](g)(-2,4),18);

0.000081415961084897 5365

>evalf(sum((-2)^7*5^4*3^k*product(7-k-j,j=0..6)*product(-k-t,t=0..3)/k!*416^((-4-k)/2)*sin((-4-k)*arctan(5)+k*Pi/2),k=0..infinity),18);

0.000081415961084897 5339
```

## IV. CONCLUSION

This article provides a new technique to calculate partial derivatives of two variables functions. We hope this technique can be applied to solve other partial differential problems. On the other hand, differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems using Maple.

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