International Journal of Scientific Research in Computer Science, Engineering and Information Technology
© 2017 IJSRCSEIT | Volume 2 | Issue 5 | ISSN : 2456-3307

A Note on Subadditivity and Antisymmetricity Involving Generalized Convex Functions

[1]R.K. Sahu, [2]B.C. Dash, [3]P.C. Nayak, [4]D.K. Dalai

[1]Department of Mathematics, Vyasanagar Autonomous College, Jajpur Road, Jajpur, Odisha, India
[2]Department of Mathematics, Choudwar College, Choudwar, Cuttack, Odisha, India
[3]Department of Mathematics, Autonomous College, Bhadrak
[4]Principal, Govt. Junior College, Ayeba, Kendrapada, Dist – Kendrapada, Odisha, India

ABSTRACT

J.Sandor [1], [2], [3], [4] has studied the η-convex function which he first defined in 1988. He also introduced and studied η-invexity. In 2003 J. Sandor [5] introduced the notion of A- convexity. In this chapter we have generalized notion of B-subadditivity, antisymmetricity,A-convexity and η- invexityand studied their properties in different theorems.

Keywords : B -subadditive function, η-invex, A-convex, C-increasing, B-decreasing, B-subadditive, η-A-invex.

2010 Mathematics Subject Classification: 62P10, 62DXX, 60A86

DEFINITION - 1: CONVEX FUNCTION:

Let V be a vector space and B : V6 $\to$ R be a map.
Then the function: V6 $\to$ R is called B-subadditive (super additive), if
f (x1 + x2 + x3 + x4 + x5 + x6) ≤ (≥)f(x1) + f (x2) + f (x3) + f (x4) + f (x5) + f (x6)
+ B(x1 , x2 , x3 , x4 , x5 , x6) for all xi ∈ V, i = 1, 2, 3, 4, 5, 6.

DEFINITION - 2: ANTISYMMETRIC:

The map B : V6 $\to$ R is called antisymmetric if
B(x1 , x2 , x3 , x4 , x5 , x6)= –B (xσ(1) , xσ(2) , xσ(3) , xσ(4) , xσ(5) , xσ(6) and B (x1 , x2 , x3 , x4 , x5 , x6) = –B(xτ(1) , xτ(2) , xτ(3) , xτ(4) , xτ(4) , xτ(5) , xτ(6)).

Here σ and τ denote even and odd permutations respectively.

THEOREM - 1:

If B is antisymmetric map and f is B-suadditive (superadditive), then f is subadditive (super additive).

Proof: As f is B-subadditive (super additive) we can write.

$$f\left(\sum_{i=1}^{6}x_i\right)\le(\ge)\sum_{i=1}^{6}f\left(x_i\right)+\left(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)},x_{\sigma(6)}\right) \qquad (1)$$

and

$$f\left(\sum_{i=1}^{6}x_i\right)\le(\ge)\sum_{i=1}^{6}f\left(x_i\right)+\left(x_{\tau(1)},x_{\tau(2)},x_{\tau(3)},x_{\tau(4)},x_{\tau(5)},x_{\tau(6)}\right) \qquad (2)$$

Adding all in equations in (1) over even permutations we have

$$\sum_{\sigma}f\left(\sum_{i=1}^{6}x_i\right)\le(\ge)\sum_{\sigma}\left(\sum_{i=1}^{6}x_i\right)+\sum_{\sigma}B\left(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)},x_{\sigma(6)}\right)$$

$$\Rightarrow f\left(\sum_{i=1}^{6}x_i\right)\sum_{\sigma}1\le(\ge)\left(\sum_{i=1}^{6}x_i\right)\sum_{\sigma}1+\sum_{\sigma}B\left(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)},x_{\sigma(6)}\right)$$

$$\Rightarrow \frac{6!}{2}f\left(\sum_{i=1}^{6}x_i\right)\le(\ge)\frac{6!}{2}\left(\sum_{i=1}^{6}f(x_i)\right)+\sum_{\sigma}B\left(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)},x_{\sigma(6)}\right) \qquad (3)$$

Adding all in equations in (2) over odd permutations we have

$$\sum_{\tau}f\left(\sum_{i=1}^{6}x_i\right)\le(\ge)\sum_{\tau}\left(\sum_{i=1}^{6}x_i\right)+\sum_{\tau}B\left(x_{\tau(1)},x_{\tau(2)},x_{\tau(3)},x_{\tau(4)},x_{\tau(5)},x_{\tau(6)}\right)$$

$$\Rightarrow f\left(\sum_{i=1}^{6}x_i\right)\sum_{\tau}1\le(\ge)\sum_{\tau}\sum_{i=1}^{6}f(x_i)\sum_{\tau}1+\sum_{\tau}B\left(x_{\tau(1)},x_{\tau(2)},x_{\tau(3)},x_{\tau(4)},x_{\tau(5)},x_{\tau(6)}\right)$$

$$\Rightarrow \frac{6!}{2} f\left(\sum_{i=1}^{6} x_i\right) \le (\ge) \frac{6!}{2}\sum_{i=1}^{6} f(x_i) + \sum_{\tau} B\left(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)}\right) \quad (4)$$

Adding (3) and (4) we have

$720 f\left(\sum_{i=1}^{6} x_i\right) \le (\ge) 720 \sum_{i=1}^{6} f(x_i)$ as all other items are cancelled due to odd and even permutations.

Hence $f\left(\sum_{i=1}^{6} x_i\right) \le (\ge) \sum_{i=1}^{6} f(x_i)$

$\Rightarrow$ f is subadditive (super additive)

DEFINITION - 3: ABSOLUTELY B-SUBADDITIVE

Let B: V6 $\rightarrow$ R be a map with vector space V. Then f : V $\rightarrow$ R be absolutely B-subadditive. If

$$\left| f\left(\sum_{i=1}^{6} x_i\right) - \sum_{i=1}^{6} f(x_i)\right| \le B\left(x_1, x_2, x_3, x_4, x_5, x_6\right)$$

for all $x_i \in V, \quad i = 1,2,3,4,5,6$

THEOREM - 2:

Let B : V6 $\rightarrow$ Rbe a map and f : V $\rightarrow$ R be absolutely B-subadditive. Then there is an additive function g: V $\rightarrow$ R such that

$|f(x) - g(x)| \le B(x_1, x_2, x_3, x_4, x_5, x_6)$

Proof : Let

$x_1 = 6^{n-1}x, x_2 = 6^{n-1}x, x_3 = 6^{n-1}, x_4 = 6^{n-1}x, x_5 = 6^{n-1}, x_6 = 6^{n-1}x,$

Since f is absolutely B-Subadditive, we can write

$$\left| f\left(\sum_{i=1}^{6} x_i\right) - \sum_{i=1}^{6} f(x_i)\right| \le B\left(x_1, x_2, x_3, x_4, x_5, x_6\right)$$

$$\Rightarrow \left| f\left(6\times 6^{n-1}x\right) - 6f\left(6^{n-1}x\right)\right| \le B\left(6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x,\right)$$

$$\Rightarrow \left| f6^n x - 6f\left(6^{n-1}x\right)\right| \le B\left(x,x,x,x,x,x,\right)$$

$$\Rightarrow \left| \frac{f\left(6^n x\right)}{6^n} - \frac{6f\left(6^{n-1}x\right)}{6^n}\right| \le B\frac{(x,x,x,x,x,x)}{6^n}$$

$$\Rightarrow \left| \frac{f\left(6^n x\right)}{6^n} - \frac{f\left(6^{n-1}x\right)}{6^{n-1}}\right| \le B\frac{(x,x,x,x,x,x)}{6^n}$$

For n>m

$$\left|\frac{f(6^n x)}{6^n} - \frac{f(6^m x)}{6^m}\right| \le \left|\frac{f(6^n x)}{6^n} - \frac{f(6^{n-1} x)}{6^{n-1}}\right| + \left|\frac{f(6^{n-1} x)}{6^{n-1}} - \frac{f(6^{n-2} x)}{6^{n-2}}\right| + \ldots + \left|\frac{f(6^{m+1} x)}{6^{m+1}} - \frac{f(6^m x)}{6^m}\right|$$

$$\le B\left(x,x,x,x,x,x,\right)\left(\frac{1}{6^n} + \frac{1}{6^{n-1}} + \ldots + \frac{1}{6^m}\right)$$

It shows that the sequence with general term $\left(x_n\right) = \left|\frac{f\left(6^n x\right)}{6^n}\right|$ is Cauchy, which converges.

Let $\lim_{n\to\infty} \frac{f\left(6^n x\right)}{6^n} = g\left(x\right)$ (say)

Hence $\left| g\left(\sum_{i=1}^{6} x_i\right) - \sum_{i=1}^{6} g\left(x_i\right)\right|$

$$= \lim_{n\to\infty}\left| \frac{f\left(\sum_{i=1}^{6} x_i\right)}{6^n} - \sum_{i=1}^{6}\frac{f\left(6^n x\right)}{6^n}\right| \le \lim_{n\to\infty} \frac{B\left(x_1, x_2, x_3, x_4, x_5, x_6\right)}{6^n} = 0$$

$$\Rightarrow g\left(\sum_{i=1}^{6} x_i\right) = \sum_{i=1}^{6} g\left(x_i\right)$$

$\Rightarrow$ g is an additive map.

To prove the uniqueness of g.

Suppose that there exists another function h such that |f (x) - h(x)| $\le$ B (x, x, x, x, x, x) and |f (x) - g(x)| $\le$ B (x, x, x, x, x, x)

Then we have |g(6^nx) - h(6^n x)| $\ge$ 2B (x, x, x, x, x, x)

$$\Rightarrow \left| g\left(6^n x\right) - h\left(6^n x\right)\right| \le 2B\left(6^n x, 6^n x, 6^n x, 6^n x, 6^n x, 6^n x\right)$$

$$\Rightarrow \left| g\left(x\right) - h\left(x\right)\right| \le \frac{2B\left(x,x,x,x,x,x\right)}{6^n} \rightarrow 0 \text{ as } n\rightarrow\infty$$

$$\left(\begin{array}{l} \because \quad = 6^n g\left(x\right) \\ \text{and } h\left(6^n x\right) = 6^n h\left(x\right) \end{array}\right)$$

$\Rightarrow g(x) = h(x)$ for all x. Hence g is unique.

By induction we have

$$\left| f\left(6^n x\right) - 6^n\left(f(x)\right)\right| \le 6^n B\left(x,x,x,x,x,x\right)$$

$$\left| g(x) - h(x)\right| \le B\left(x,x,x,x,x,x\right)$$

DEFINITION-4: B-DECREASING

The Map $g: R_+ \rightarrow R$ is called B-decreasing on R_+ if

$$x_1 > x_2 \Rightarrow g\left(x_1\right) \le g\left(x_2\right) + B\left(x_1, x_2, x_3, x_4, x_5, x_6\right)$$

for all $x_1 \in R_+$, i=1,2,3,4,5,6.

THEOREM-3:

Let $f: R_+ \rightarrow R$ be a function such that the map

$x \rightarrow \frac{f(x)}{x}$ is B-decreasing on R_+.

Then f is B_1 –subadditive, where

$$B_1(x_1,x_2,x_3,x_4,x_5,x_6)=x_1.B\left(\sum_{i=1}^{6}x_i,x_1,x_1,x_1,x_1,x_1\right)$$

$$+x_2.B\left(\sum_{i=1}^{6}x_i,x_2,x_2,x_2,x_2,x_2\right)$$

$$+x_3.B\left(\sum_{i=1}^{6}x_i,x_3,x_3,x_3,x_3,x_3\right)+x_4.B\left(\sum_{i=1}^{6}x_i,x_4,x_4,x_4,x_4,x_4\right)$$

$$+x_5.B\left(\sum_{i=1}^{6}x_i,x_5,x_5,x_5,x_5,x_5\right)+x_6.B\left(\sum_{i=1}^{6}x_i,x_6,x_6,x_6,x_6,x_6\right)$$

for all $x\in R_+, i=1,2,3,4,5,6$

Proof: Since $x_i\in R_+, i=1,2,3,4,5,6.$

we have $\sum_{i=1}^{6}x_i>x_1\Rightarrow\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}>x_1$

$$\Rightarrow\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_1)}{x_1}+\left(\sum_{i=1}^{6}x_{i,}x_1,x_1,x_1,x_1,x_1\right)$$

Similarly, we obtain

$$\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_2)}{x_2}+\left(\sum_{i=1}^{6}x_{i,}x_2,x_2,x_2,x_2,x_2\right),$$

$$\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_3)}{x_3}+\left(\sum_{i=1}^{6}x_{i,}x_3,x_3,x_3,x_3,x_3\right),$$

$$\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_4)}{x_4}+\left(\sum_{i=1}^{6}x_{i,}x_4,x_4,x_4,x_4,x_4\right),$$

$$\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_5)}{x_5}+\left(\sum_{i=1}^{6}x_{i,}x_5,x_5,x_5,x_5,x_5\right)$$

$$\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}\le\frac{f(x_6)}{x_6}+\left(\sum_{i=1}^{6}x_{i,}x_6,x_6,x_6,x_6,x_6\right)$$

So we have,

$$f(x_1+x_2+x_3+x_4+x_5+x_6)=\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}\left(\sum_{i=1}^{6}x_i\right)$$

$$=\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_1+\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_2+\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_3+\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_4+\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_5+\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}x_6$$

$$\le\sum_{i=1}^{6}f(x_i)+x_1B\left(\sum_{i=1}^{6}x_{i,}x_1,x_1,x_1,x_1,x_1\right)+x_2B\left(\sum_{i=1}^{6}x_{i,}x_2,x_2,x_2,x_2,x_2\right)+x_3.B\left(\sum_{i=1}^{6}x_{i,}x_3,x_3,x_3,x_3,x_3\right)$$

$$+x_4B\left(\sum_{i=1}^{6}x_i,x_4,x_4,x_4,x_4,x_4\right)+x_5B\left(\sum_{i=1}^{6}x_i,x_5,x_5,x_5,x_5,x_5\right)+x_6B\left(\sum_{i=1}^{6}x_i,x_6,x_6,x_6,x_6,x_6\right)$$

$\Rightarrow$ f is B_1 –subadditive.

DEFINITION -5: A-CONVEX

Let C be a convex subset of a vector space V. Let A: $C\times C\times C\to R$ be a function of 3 variables. The function $f:C\to R$ is called A-convex (concave) if the following relation holds

$$f(\lambda x+(1-\lambda)y)\le(\ge)\lambda f(x)+(1-\lambda)f(y)+\lambda(x-y).A(\lambda x+(1-\lambda)y,x,y)$$

for all $x,y\in C$ and $0\le\lambda\le1$.

THEOREM-4:

Let $A:R_+\times R_+\times R_+\to R$ and $f:R_+\to R$ be A-concave function with $A:C\times C\to R$ where C is convex vector space.

Let $A_0(0,0)=A(0,0,0)$

Assume that $f(0)=0$

Then f is B_1-subadditive function, where

$$B_1(x_1,x_2,x_3,x_4,x_5,x_6)-x_1A_1\left(x_1,\sum_{i=1}^{6}x_i\right)-x_2A_1\left(x_2,\sum_{i=1}^{6}x_i\right)$$

$$-x_3A_1\left(x_3,\sum_{i=1}^{6}x_i\right)-x_4A_1\left(x_4,\sum_{i=1}^{6}x_i\right)-x_5A_1\left(x_5,\sum_{i=1}^{6}x_i\right)-x_6A_1\left(x_6,\sum_{i=1}^{6}x_i\right)$$

Proof: Recall that the function f is A-convex (A-concave) if

$$\frac{f(x)-f(z)}{x-z}\le(\ge)\frac{f(y)-f(z)}{y-z}+A(x,y,z)\text{ with }x<z<y.$$

By hypothesis f is A-Concave, then one can write,

$$\frac{f(x)-f(z)}{x-z}\le\frac{f(y)-f(z)}{y-z}+A(x,y,z)$$

$$\Rightarrow(y-z)(f(x)-f(z))\ge(x-z)(f(y)-f(z))+(y-z)(x-z)A(x,y,z)$$

$$\Rightarrow (y-z)f(x)-(y-z)f(z)\geq (x-z)f(y)-(x-z)f(z)+(y-z)(x-z)A(x,y,z)$$

$$\Rightarrow (y-z)f(x)\geq (x-z)f(y)+(y-z-x+z)f(z)+(y-z)(x-z)A(x,y,z)$$

$$\Rightarrow (y-z)f(x)\geq (x-z)f(y)+(y-x)f(z)+(y-z)(x-z)A(x,y,z)$$

$$\Rightarrow f(x)\geq \left(\frac{x-z}{y-z}\right)f(y)+\left(\frac{y-x}{y-z}\right)f(z)+(x-z)A(x,y,z)$$

Take $\lambda=\dfrac{x-z}{y-z}\in(0,1)$

$$\Rightarrow 1-\lambda=\frac{y-x}{y-z}$$

Also $\lambda y+(1-\lambda)z=\dfrac{x-z}{y-z}y+\dfrac{y-x}{y-z}z=x$

Now $f(x)\geq \lambda f(y)+(1-\lambda)f(z)+(x-z)A(x,y,z)$

By assumption f(0) =0

We have, $\dfrac{f(x)-f(0)}{x-0}=\dfrac{f(x)}{x}$

Now from above, w observe that the function $\dfrac{f(x)}{x}$ is A_1-increasing.

So one can write

$$\frac{f(x_1)}{x}\geq \frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_1,\sum_{i=1}^{6}x_i\right),$$

$$\frac{f(x_2)}{x_2}\geq \frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_2,\sum_{i=1}^{6}x_i\right),$$

$$\frac{f(x_3)}{x}\geq \frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_3,\sum_{i=1}^{6}x_i\right),$$

$$\frac{f(x_4)}{x_4}\geq \frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_4,\sum_{i=1}^{6}x_i\right),$$

$$\frac{f(x_5)}{x_5}\geq \frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_5,\sum_{i=1}^{6}x_i\right),$$

and $\dfrac{f(x_6)}{x_6}\geq \dfrac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}(x_i)}+A_1\left(x_6,\sum_{i=1}^{6}x_i\right).$

On simplification one can write,

$$\sum_{i=1}^{6}f(x_i)\geq x_1\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}+x_2\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}+x_3\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}$$

$$+x_4\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}+x_5\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}+x_6\frac{f\left(\sum_{i=1}^{6}x_i\right)}{\sum_{i=1}^{6}x_i}$$

$$=x_1A_1\left(x_1,\sum_{i=1}^{6}x_i\right)+x_2A_1\left(x_2,\sum_{i=1}^{6}x_i\right)+x_3A_1\left(x_3,\sum_{i=1}^{6}x_i\right)+x_4A_1\left(x_4,\sum_{i=1}^{6}x_i\right)$$

$$+x_5A_1\left(x_5,\sum_{i=1}^{6}x_i\right)+x_6A_1\left(x_6,\sum_{i=1}^{6}x_i\right)$$

$$=f\left(\sum_{i=1}^{6}x_i\right)-B_1(x_1,x_2,x_3,x_4,x_5,x_6)$$

$$=f\left(\sum_{i=1}^{6}x_i\right)\leq \sum_{i=1}^{6}f(x_1)+B_1(x_1,x_2,x_3,x_4,x_5,x_6)$$

$\Rightarrow$ f isB_1-subadditive with given

$$B_1(x_1,x_2,x_3,x_4,x_5,x_6)=-x_1A_1\left(x_1,\sum_{i=1}^{6}x_i\right)-x_2A_1\left(x_2,\sum_{i=1}^{6}x_i\right)-x_3A_1\left(x_3,\sum_{i=1}^{6}x_i\right)$$

$$-x_4A_1\left(x_4,\sum_{i=1}^{6}x_i\right)-x_5A_1\left(x_5,\sum_{i=1}^{6}x_i\right)-x_6A_1\left(x_6,\sum_{i=1}^{6}x_i\right)$$

THEOREM -5:

Let $f:R_+\rightarrow R$ be convex and B-subadditive

Then function is C-increasing with $C:R_+\times R_+\rightarrow R$

Proof: Let $\lambda=\dfrac{5x}{5x+h}\in(0,1),h>0$

$$\therefore\ \lambda x+(1-\lambda)(6x+h)=\frac{5x}{5x+h}.x+\left(1-\frac{5x}{5x+h}\right)(6x+h)$$

$$=\frac{5x^2}{5x+h}+\frac{h}{5x+h}(6x+h)$$

$$=\frac{5x^2}{5x+h}+\frac{6xh+h^2}{5x+h}$$

$$=\frac{5x^2+6xh+h^2}{5x+h}$$

$$=\frac{(x+h)(5x+h)}{5x+h}=x+h$$

Since f is B-subadditive, we have

$$f(6x+h)=f(x+x+x+x+x+x+h)$$

$$\le f(x)+f(x)+f(x)+f(x)+f(x)+f(x+h)+B(x,x,x,x,x,x+h)$$

$$=5f(x)+f(x+h)+B(x,x,x,x,x,x+h)$$

Since f is convex,

$$f(x+h)\le\lambda f(x)+(1-\lambda)f(x+h)$$

$$\le\lambda f(x)(1-\lambda)[5f(x)+f(x+h)+B(x,x,x,x,x,x+h)]$$

$$=\lambda f(x)+5f(x)-5\lambda f(x)+f(x+h)-\lambda f(x+h)+(1-\lambda)B(x,x,x,x,x,x+h)$$

$$\Rightarrow\lambda f(x+h)\le 5f(x)-4\lambda f(x)+(1-\lambda)B(x,x,x,x,x,x+h)$$

$$\Rightarrow\frac{5x}{5x+h}f(x+h)\le 5f(x)-4.\frac{5x}{5x+h}f(x)+\left(1-\frac{5x}{5x+h}\right)B(x,x,x,x,x,x+h)$$

$$\Rightarrow\frac{5x}{5x+h}f(x+h)\le\frac{5(5x+h)f(x)-20xf(x)}{5x+h}+\frac{5x}{5x+h}B(x,x,x,x,x,x+h)$$

$$\Rightarrow\frac{5x}{5x+h}f(x+h)\le\frac{5(5x+h)}{5x+h}f(x)+\frac{h}{5x+h}B(x,x,x,x,x,x+h)$$

$$\Rightarrow 5xf(x+h)\le 5(5x+h)f(x)+h.B(x,x,x,x,x,x+h)$$

$$\Rightarrow\frac{5xf(x+h)}{5x(x+h)}\le\frac{f(x)}{x}+\frac{h.B(x,x,x,x,x,x+h)}{5x(x+h)}$$

$$\Rightarrow\frac{f(x+h)}{(x+h)}\le\frac{f(x)}{x}+C(x,h)\ \text{where}$$

$$C(x,y)=\frac{h.B(x,x,x,x,x,x+h)}{5x(x+h)}$$

Then function $\frac{f(x)}{x}$ is C-increasing function.

2. $\wedge-$INVEX FUNCTION:

Definition 2.1: Let V be a vector space and $S\subset V$ be η invex subset of V where $\eta:V^6\to V$ and $f:S\to R_\infty$ where $R_\infty=R\cup\{\infty\}$. Then f is called $\eta-\wedge$-invex function if

$$f(p+\lambda\eta(p,q,r,s,t,u))\le\max\{f(p),f(q),f(r),f(s),f(t),f(u)\}$$

for all p, q, r, s, t, u $\in S$ and $\lambda\in\wedge$

DEFINITION 2.2

The set D(f) = { $p\in S:f(p)<\infty$ } is called effective domain of $f:S\to R_+$.

DEFINITION 2.3

The point $p\in S$ with f(p) = ∞ is called a singular point of f. The set of all singular points of f is denoted by S(f).

THEOREM 2.1

Let $f:V\to R_\infty$ be $\eta-\wedge$-invex set and Let $K\subset D$ (f) be an open η-invex set.

Let us assume that $\eta:V^6\to V$ be continuous and $f(p)>-\infty$ for all $p\in V$.

Then the function $f:K\to R$ is η-Crazi-inve.

Proof: Let p, q, r, s, t, u $\in K$. Then there exist a, b, c $\in(0,1)$ with

$$x=p+a\eta(p,q,r,s,t,u)\in K$$

$$y=q+b\eta(p,q,r,s,t,u)\in K$$

$$z=r+c\eta(p,q,r,s,t,u)\in K$$

Since we are in case of normed space, we can select sequences (p_k), (q_k), (r_k), (s_k), (t_k) and (u_k). such that $p_k\to p, q_k\to q, r_k\to r, s_k\to s, t_k\to t$ and $u_k\to u$ as $k\to\infty$

$$\Rightarrow f(p_k)\to\underline{f}(p), f(q_k)\to\underline{f}(q), f(r_k)\to\underline{f}(r)$$

$$f(s_k)\to\underline{f}(s), f(t_k)\to\underline{f}(t)\text{ and }f(u_k)\to\underline{f}(u)\text{ as }k\to\infty$$

Let $a_k,b_k,c_k\in\wedge$ be sequences such that $a_k\to a, b_k\to b$ and $c_k\to c$.

Put $x_k=p_k+a_k\eta(p_k,q_k,r_k,s_k,t_k,u_k)$

$$y_k=q_k+b_k\eta(p_k,q_k,r_k,s_k,t_k,u_k)$$

$$z_k=r_k+c_k\eta(p_k,q_k,r_k,s_k,t_k,u_k)$$

Then we have

$$x_k\to p+a\eta(p,q,r,s,t,u)=x$$

$$y_k \to q + b\eta(p,q,r,s,t,u) = y$$
$$z_k \to r + c\eta(p,q,r,s,t,u) = z$$

But $\underline{f}(x) \le \lim_{k\to\infty}$ inf $f(x_k)$ and

$$f(x_k \le) \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

$$\underline{f}(y) \le \lim_{k\to\infty} \inf f(y_k) \text{ and } f(y_k) \le \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

$$\underline{f}(z) \le \lim_{k\to\infty} \inf f(z_k) \text{ and } f(z_k) \le \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

Taking the limit as $k \to \infty$ we get

$$\underline{f}(x) \le \lim_{k\to\infty} \inf f(x_k) \le \max\left\{\lim_{k\to\infty} f(p_k). \lim_{k\to\infty} f(q_k). \lim_{k\to\infty} f(r_k). \lim_{k\to\infty} f(s_k). \lim_{k\to\infty} f(t_k). \lim_{k\to\infty} f(u_k).\right\}$$

$$= \max\{\underline{f}(p), \underline{f}(q), \underline{f}(r), \underline{f}(s), \underline{f}(t), \underline{f}(u)\}$$

$$\underline{f}(y) \le \lim_{k\to\infty} \inf f(y_k) \le \max\left\{\lim_{k\to\infty} f(p_k). \lim_{k\to\infty} f(q_k). \lim_{k\to\infty} f(r_k). \lim_{k\to\infty} f(s_k). \lim_{k\to\infty} f(t_k). \lim_{k\to\infty} f(u_k).\right\}$$

= max {(p), f(q), f(r), f(s), f(t), f(u)}

$$\underline{f}(z) \le \lim_{k\to\infty} \inf f(z_k) \le \max\left\{\lim_{k\to\infty} f(p_k). \lim_{k\to\infty} f(q_k). \lim_{k\to\infty} f(r_k). \lim_{k\to\infty} f(s_k). \lim_{k\to\infty} f(t_k). \lim_{k\to\infty} f(u_k).\right\}$$

$$= \max\{\underline{f}(p), \underline{f}(q), \underline{f}(r), \underline{f}(s), \underline{f}(t), \underline{f}(u)\}$$

$\therefore$ f is $\eta-$Crazi-invex.

Proposition 2.1: If $f: X \to R_\infty$ is $\eta-$ nvex (or η-Crazi-invex) then the set D(f) is η-invex set (or η-Crazi-invex set)

Proof: Let p, q, r, s, t, u $\in D(f)$

Then

$$f(p) < +\infty, f(q) < +\infty, f(r) < +\infty, f(s) < +\infty, f(t) < +\infty, f(u) < +\infty.$$

Then

$$f(p + \lambda\eta(p,q,r,s,t,u)) \le \lambda f(q) + (1-\lambda) f(r) < +\infty$$

(in η-invex case)

$$\Rightarrow f(p + \lambda\eta(p,q,r,s,t,u) \le \max\{f(p), f(q), f(r), f(s), f(t), f(u)\} < \infty$$

In any case $p + \lambda\eta(p,q,r,s,t,u) \in D(f)$

$\Rightarrow D(f)$ is $\eta-$invex.

THEOREM 2.2

Let V be a real banach space and η be a function such that for $M \subset V$ is $p, p_0, q, q_0, r, r_0 \in \text{int}\, M_0$ then there exists $\lambda \in [0,1]$ and $u \in M$ such that

$$\left.\begin{aligned} p &= p_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \\ q &= q_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \\ r &= r_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \end{aligned}\right\} \qquad (*)$$

$\ldots\ldots\ldots \to R$ be $\eta - \wedge -$ invex function and let $p_0, q_0, r_0 \in \text{int}\, D(f)$ such that $\bar{f}(p_0) \le +\infty, \bar{f}(q_0) \le +\infty, \bar{f}(r_0) \le +\infty$ if η is non expansive related to the second argument then $\bar{f}(p) \le +\infty, \bar{f}(q) \le +\infty, \bar{f}(r) \le +\infty$ for all p,q, r $\in$ int D(f).

Proof: Let M = D(f) and let $p, p_0, q, q_0, r, r_0 \in D(f)$.

where $\bar{f}(p) = +\infty, \bar{f}(q) = +\infty, \bar{f}(r) = +\infty$

$\bar{f}(p_0) < +\infty, \bar{f}(q_0) < +\infty, \bar{f}(r_0) < +\infty$

By (*) and $u \in D(f)$, there exists $\lambda \in \wedge$ such that

$$p = p_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$
$$q = q_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$
$$r = r_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$

Select now sequences (p_k), (q_k), and (r_k) with $p_k D(f) - \{p\}, \in D(f) - \{q\}, r_k \in D(f) - \{r\}$ such that

$$p_k \to p, f(p_k) \to +\infty$$
$$q_k \to q, f(q_k) \to +\infty$$
$$r_k \to r, f(r_k) \to +\infty$$

There exist $k_1, k_2, k_3 \in N$ such that

$$f(p_k) > f(u) \text{ for } k \ge k_1$$
$$f(q_k) > f(u) \text{ for } k \ge k_2$$
$$f(r_k) > f(u) \text{ for } k \ge k_3$$

Let $k_0 = \max\{k_1, k_2, k_3\}$

So $f(p_k) > f(u), f(q_k) > f(u), f(r_k) > f(u)$ for all $k > k_0$

Let x_k, y_k and z_k be determined by the equations.

$$\left.\begin{aligned} p_k &= x_k + \lambda\eta(u,t,s,r,q,x_k) \\ q_k &= y_k + \lambda\eta(u,t,s,r,q,y_k) \\ r_k &= z_k + \lambda\eta(u,t,s,r,q,z_k) \end{aligned}\right\}$$

(1)

Equation (1) can be solved for all x_k, y_k and z_k (k fixed). Since by setting $x_k = x, y_k = y$ and $z_k = z$ the map.

$$g(x) = p - \lambda\eta(u,t,s,r,q,x)$$
$$g_1(y) = q - \lambda\eta(u,t,s,r,q,y)$$

$$g_2(z) = r - \lambda\eta(u,t,s,r,q,z)$$

$g, g_1, g_2 : V \to V$ becomes a contradiction.

Indeed on has

$$\|g(x_1) - g(x_2)\| = \lambda\|\eta(u,t,s,r,q,x_1) - \eta(u,t,s,r,q,x_2)\| \le \lambda < 1$$
$$\|g_1(y_1) - g_1(y_2)\| = \lambda\|\eta(u,t,s,r,q,y_1) - \eta(u,t,s,r,q,y_2)\| \le \lambda < 1$$

also

$$\|g_2(z_1) - g_2(z_2)\| = \lambda\|\eta(u,t,s,r,q,z_1) - \eta(u,t,s,r,q,z_2)\| \le \lambda < 1$$

η being non-expansive upon the third argument.

Now Banch's classical contradiction principle assures the existence of unique fixed point of operators g on g_1 and g_2.

We shall now prove that $x_k \to p_0, y_k \to q_0$ and $z_k \to r_0$

Now, $\|p_k - p\| = \|x_k - p + \lambda\eta(u,s,t,r,q,x_k)\|$

$$= \|x_k - p_0 - \lambda\eta\|(u,s,t,r_0,q_0,p_0) + \lambda\eta(u,s,t,r,q,\eta)$$

$$\ge \|x_k - p_0\| - \lambda\|\eta(u,s,t,r_0,q_0,p_0) - \eta(u,s,t,r,q,x_k)\|$$

$$\ge \|x_k - p_0\| - \lambda\|x_k - p_0\|$$

$$\ge (1-\lambda)\|x_k - p_0\|$$

$$\Rightarrow \|x_k - p_0\| \le \frac{1}{1-\lambda}\|p_k - p\| \to 0 \text{ as } k \to \infty$$

Therefore $\quad x_k \to p_0$ as $k \to \infty$

Similarly $\quad y_k \to q_0$ as $k \to \infty$

and $\quad z_k \to r_0$ as $k \to \infty$.

for $k > k_0$

$$f(u) < f(p_k) = \max\{f(x_k), f(q), f(r), f(s), f(t), f(u)\} = f(x_k)$$
$$f(u) < f(q_k) = \max\{f(y_k), f(q), f(r), f(s), f(t), f(u)\} = f(y_k)$$
$$f(u) < f(r_k) = \max\{f(z_k), f(q), f(r), f(s), f(t), f(u)\} = f(z_k)$$

Then $\quad \bar{f}(p_0) \ge \lim_{k\to\infty} f(x_k) = +\infty$

$$\bar{f}(q_0) \ge \lim_{k\to\infty} f(y_k) = +\infty$$

$$\bar{f}(r_0) \ge \lim_{k\to\infty} f(z_k) = +\infty$$

which contradicts to

$\bar{f}(p_0) < +\infty, \bar{f}(q_0) < +\infty$ and $\bar{f}(r_0) < +\infty$

Remarks: If η has the non-expansivity property upon the arguments $p_0, q_0, r_0, s_0, t_0, u_0$ i.e.

$$\|\eta(p,q,r,s,t,u) - \eta(p_0,q_0,r_0,s_0,t_0,u_0)\|$$

$$\le \|p - p_0\| + \|q - q_0\| + \|r - r_0\| + \|s - s_0\| + \|t - t_0\| + \|u - u_0\|,$$

then it is immediately seen that if $M \subset X$ as an invex set. Then int M will also be invex.

REFERENCES

[1] J. Sandor, Some Open Problems in the Theory of Functional Equations, Simpozium on Applications of Fundamental Equations in Education, Science and Industry. (4th June, 1988), Odorhelue - Seculese, Romania.

[2] J. Sandor, On the Principle of Condensation Singularities, Fourth National Conference on Mathematical Inequalities, Siblu, (30-31 Oct., 1992).

[3] J. Sandor, Generalised Invexity and Its Application in Optimization Theory, First Joint Conference on Modern Applied Analysis, Sieni (Romania), (12-17 June, 1995)].

[4] J. Sandor, On Certain Classes of Generalized Convex Functions with Application, I, Studia Univ, Babes-Bolyai, Math, 44, (1999), 73-84.

[5] J. Sandor, On Cerntain Classes of Generalized Convex Functions with Application, II, Studia Univ, ''Babes-Bolyai'', Mathmatica, XLVIII (1), (2002), 109-117.

[6] R.B. Dash, D.K. Dalai and N. Mishra, A Note on a Class Generalised Convex Functions with some Applications, International Review of Pure and Applied Mathematical (July - December) 2009, Volume 5, No. 2, PP. 243-253.

[7] B.C. Dash, and D.K. Dalai On a Class of Generalised Convex Function With Some Applications, American Journal of Mathematics and Mathematical Sciences, 4(1), January-June 2015. ISSN : 2278-0874. pp, 13-20.