

A Note on Subadditivity and Antisymmetry Involving Generalized Convex Functions.

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ABSTRACT

J.Sandor [1], [2], [3], [4] has studied the η -convex function which he first defined in 1988. He also introduced and studied η -invexity. In 2003 J. Sandor [5] introduced the notion of A-convexity. In this chapter we have generalized notion of B-subadditivity, antisymmetry, A-convexity and η -invexity and studied their properties in different theorems.

Keywords : B₁-subadditive function, η -invex, A-convex, C-increasing, B-decreasing, B-subadditive, η -A-invex.

2010 Mathematics Subject Classification: 62P10, 62DXX, 60A86

DEFINITION - 1: CONVEX FUNCTION:

Let V be a vector space and $B : V^6 \rightarrow R$ be a map.

Then the function: $V^6 \rightarrow R$ is called B-subadditive

(super additive), if

$$f(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) \leq (\geq) f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)$$

+ $B(x_1, x_2, x_3, x_4, x_5, x_6)$ for all $x_i \in V, i = 1, 2, 3, 4, 5,$

6.

DEFINITION - 2: ANTISYMMETRIC:

The map $B : V^6 \rightarrow R$ is called antisymmetric if

$$B(x_1, x_2, x_3, x_4, x_5, x_6) = -B(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)},$$

$$x_{\sigma(5)}, x_{\sigma(6)}) \text{ and } B(x_1, x_2, x_3, x_4, x_5, x_6) = -B(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)}).$$

Here σ and τ denote even and odd permutations respectively.

THEOREM - 1:

If B is antisymmetric map and f is B-subadditive

(superadditive), then f is subadditive (super additive).

Proof: As f is B-subadditive (super additive) we can write.

$$f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) \sum_{i=1}^6 f(x_i) + (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}) \quad (1)$$

and

$$f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) \sum_{i=1}^6 f(x_i) + (x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)}) \quad (2)$$

Adding all in equations in (1) over even permutations we have

$$\begin{aligned} \sum_{\sigma} f\left(\sum_{i=1}^6 x_i\right) &\leq (\geq) \sum_{\sigma} \left(\sum_{i=1}^6 f(x_i)\right) + \sum_{\sigma} B(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}) \\ &\Rightarrow f\left(\sum_{i=1}^6 x_i\right) \sum_{\sigma} 1 \leq (\geq) \left(\sum_{i=1}^6 f(x_i)\right) \sum_{\sigma} 1 + \sum_{\sigma} B(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}) \\ &\Rightarrow \frac{6!}{2} f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) \frac{6!}{2} \left(\sum_{i=1}^6 f(x_i)\right) + \sum_{\sigma} B(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}) \end{aligned} \quad (3)$$

Adding all in equations in (2) over odd permutations we have

$$\begin{aligned} \sum_{\tau} f\left(\sum_{i=1}^6 x_i\right) &\leq (\geq) \sum_{\tau} \left(\sum_{i=1}^6 f(x_i)\right) + \sum_{\tau} B(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)}) \\ &\Rightarrow f\left(\sum_{i=1}^6 x_i\right) \sum_{\tau} 1 \leq (\geq) \sum_{\tau} \left(\sum_{i=1}^6 f(x_i)\right) \sum_{\tau} 1 + \sum_{\tau} B(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)}) \end{aligned}$$

$$\Rightarrow \frac{6!}{2} f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) \frac{6!}{2} \sum_{i=1}^6 f(x_i) + \sum_{\tau} B(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}, x_{\tau(6)})$$

(4)

Adding (3) and (4) we have

$$720f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) 720 \sum_{i=1}^6 f(x_i) \text{ as all other items are cancelled due to odd and even permutations.}$$

$$\text{Hence } f\left(\sum_{i=1}^6 x_i\right) \leq (\geq) \sum_{i=1}^6 f(x_i)$$

$\Rightarrow f$ is subadditive (super additive)

DEFINITION - 3: ABSOLUTELY B-SUBADDITIVE

Let $B: V^6 \rightarrow R$ be a map with vector space V . Then $f: V \rightarrow R$ be absolutely B-subadditive. If

$$\left| f\left(\sum_{i=1}^6 x_i\right) - \sum_{i=1}^6 f(x_i) \right| \leq B(x_1, x_2, x_3, x_4, x_5, x_6)$$

for all $x_i \in V, i=1,2,3,4,5,6$

THEOREM - 2:

Let $B: V^6 \rightarrow R$ be a map and $f: V \rightarrow R$ be absolutely B-subadditive. Then there is an additive function $g: V \rightarrow R$ such that

$$|f(x) - g(x)| \leq B(x_1, x_2, x_3, x_4, x_5, x_6)$$

Proof : Let

$$x_1 = 6^{n-1}x, x_2 = 6^{n-1}x, x_3 = 6^{n-1}x, x_4 = 6^{n-1}x, x_5 = 6^{n-1}x, x_6 = 6^{n-1}x,$$

Since f is absolutely B-Subadditive, we can write

$$\left| f\left(\sum_{i=1}^6 x_i\right) - \sum_{i=1}^6 f(x_i) \right| \leq B(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\Rightarrow |f(6 \times 6^{n-1}x) - 6f(6^{n-1}x)| \leq B(6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x, 6^{n-1}x)$$

$$\Rightarrow |f(6^n x) - 6f(6^{n-1}x)| \leq B(x, x, x, x, x, x)$$

$$\Rightarrow \left| \frac{f(6^n x)}{6^n} - \frac{6f(6^{n-1}x)}{6^n} \right| \leq B\left(\frac{x, x, x, x, x, x}{6^n}\right)$$

$$\Rightarrow \left| \frac{f(6^n x)}{6^n} - \frac{f(6^{n-1}x)}{6^{n-1}} \right| \leq B\left(\frac{x, x, x, x, x, x}{6^n}\right)$$

For $n > m$

$$\left| \frac{f(6^n x)}{6^n} - \frac{f(6^m x)}{6^m} \right| \leq \left| \frac{f(6^n x)}{6^n} - \frac{f(6^{n-1}x)}{6^{n-1}} \right| + \left| \frac{f(6^{n-1}x)}{6^{n-1}} - \frac{f(6^{n-2}x)}{6^{n-2}} \right| + \dots + \left| \frac{f(6^{m+1}x)}{6^{m+1}} - \frac{f(6^m x)}{6^m} \right|$$

$$\leq B(x, x, x, x, x, x) \left(\frac{1}{6^n} + \frac{1}{6^{n-1}} + \dots + \frac{1}{6^m} \right)$$

It shows that the sequence with general term

$$(x_n) = \left| \frac{f(6^n x)}{6^n} \right| \text{ is Cauchy, which converges.}$$

$$\text{Let } \lim_{n \rightarrow \infty} \frac{f(6^n x)}{6^n} = g(x) \text{ (say)}$$

$$\text{Hence } \left| g\left(\sum_{i=1}^6 x_i\right) - \sum_{i=1}^6 g(x_i) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{f\left(\sum_{i=1}^6 x_i\right)}{6^n} - \sum_{i=1}^6 \frac{f(6^n x)}{6^n} \right| \leq \lim_{n \rightarrow \infty} \frac{B(x_1, x_2, x_3, x_4, x_5, x_6)}{6^n} = 0$$

$$\Rightarrow g\left(\sum_{i=1}^6 x_i\right) = \sum_{i=1}^6 g(x_i)$$

$\Rightarrow g$ is an additive map.

To prove the uniqueness of g .

Suppose that there exists another function h such that $|f(x) - h(x)| \leq B(x, x, x, x, x, x)$ and $|f(x) - g(x)| \leq B(x, x, x, x, x, x)$

Then we have $|g(6^n x) - h(6^n x)| \geq 2B(x, x, x, x, x, x)$

$$\Rightarrow |g(6^n x) - h(6^n x)| \leq 2B(6^n x, 6^n x, 6^n x, 6^n x, 6^n x, 6^n x)$$

$$\Rightarrow |g(x) - h(x)| \leq \frac{2B(x, x, x, x, x, x)}{6^n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\left(\begin{array}{l} \because g(6^n x) = 6^n g(x) \\ \text{and } h(6^n x) = 6^n h(x) \end{array} \right)$$

$\Rightarrow g(x) = h(x)$ for all x . Hence g is unique.

By induction we have

$$\left| f(6^n x) - 6^n f(x) \right| \leq 6^n B(x, x, x, x, x, x)$$

$$|g(x) - h(x)| \leq B(x, x, x, x, x, x)$$

DEFINITION-4: B-DECREASING

The Map $g: R_+ \rightarrow R$ is called B-decreasing on R_+ if

$$x_1 > x_2 \Rightarrow g(x_1) \leq g(x_2) + B(x_1, x_2, x_3, x_4, x_5, x_6)$$

for all $x_i \in R_+, i=1,2,3,4,5,6$.

THEOREM-3:

Let $f: R_+ \rightarrow R$ be a function such that the map

$$x \rightarrow \frac{f(x)}{x} \text{ is B-decreasing on } R_+.$$

Then f is B_1 -subadditive, where

$$\begin{aligned}
& B_1(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 \cdot B\left(\sum_{i=1}^6 x_i, x_1, x_1, x_1, x_1, x_1\right) \\
& + x_2 \cdot B\left(\sum_{i=1}^6 x_i, x_2, x_2, x_2, x_2, x_2\right) \\
& + x_3 \cdot B\left(\sum_{i=1}^6 x_i, x_3, x_3, x_3, x_3, x_3\right) + x_4 \cdot B\left(\sum_{i=1}^6 x_i, x_4, x_4, x_4, x_4, x_4\right) \\
& + x_5 \cdot B\left(\sum_{i=1}^6 x_i, x_5, x_5, x_5, x_5, x_5\right) + x_6 \cdot B\left(\sum_{i=1}^6 x_i, x_6, x_6, x_6, x_6, x_6\right)
\end{aligned}$$

for all $x \in \mathbb{R}_+, i = 1, 2, 3, 4, 5, 6$

Proof: Since $x_i \in \mathbb{R}_+, i = 1, 2, 3, 4, 5, 6$.

$$\begin{aligned}
\text{we have } \sum_{i=1}^6 x_i > x_1 & \Rightarrow \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} > x_1 \\
& \Rightarrow \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} \leq \frac{f(x_1)}{x_1} + B\left(\sum_{i=1}^6 x_i, x_1, x_1, x_1, x_1, x_1\right)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} & \leq \frac{f(x_2)}{x_2} + B\left(\sum_{i=1}^6 x_i, x_2, x_2, x_2, x_2, x_2\right), \\
\frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} & \leq \frac{f(x_3)}{x_3} + B\left(\sum_{i=1}^6 x_i, x_3, x_3, x_3, x_3, x_3\right), \\
\frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} & \leq \frac{f(x_4)}{x_4} + B\left(\sum_{i=1}^6 x_i, x_4, x_4, x_4, x_4, x_4\right), \\
\frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} & \leq \frac{f(x_5)}{x_5} + B\left(\sum_{i=1}^6 x_i, x_5, x_5, x_5, x_5, x_5\right) \\
\frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} & \leq \frac{f(x_6)}{x_6} + B\left(\sum_{i=1}^6 x_i, x_6, x_6, x_6, x_6, x_6\right)
\end{aligned}$$

So we have,

$$\begin{aligned}
f\left(x_1 + x_2 + x_3 + x_4 + x_5 + x_6\right) & = \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} \left(\sum_{i=1}^6 x_i\right) \\
& = \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_1 + \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_2 + \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_3 + \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_4 + \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_5 + \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} x_6 \\
& \leq \sum_{i=1}^6 f(x_i) + x_1 B\left(\sum_{i=1}^6 x_i, x_1, x_1, x_1, x_1, x_1\right) + x_2 B\left(\sum_{i=1}^6 x_i, x_2, x_2, x_2, x_2, x_2\right) + x_3 B\left(\sum_{i=1}^6 x_i, x_3, x_3, x_3, x_3, x_3\right) \\
& \quad + x_4 B\left(\sum_{i=1}^6 x_i, x_4, x_4, x_4, x_4, x_4\right) + x_5 B\left(\sum_{i=1}^6 x_i, x_5, x_5, x_5, x_5, x_5\right) + x_6 B\left(\sum_{i=1}^6 x_i, x_6, x_6, x_6, x_6, x_6\right)
\end{aligned}$$

$\Rightarrow f$ is B_1 -subadditive.

DEFINITION -5: A-CONVEX

Let C be a convex subset of a vector space V . Let $A: C \times C \times C \rightarrow \mathbb{R}$ be a function of 3 variables. The function $f: C \rightarrow \mathbb{R}$ is called A-convex (concave) if the following relation holds

$$f(\lambda x + (1-\lambda)y) \leq (\geq) \lambda f(x) + (1-\lambda)f(y) + \lambda(x-y)A(\lambda x + (1-\lambda)y, x, y)$$

for all $x, y \in C$ and $0 \leq \lambda \leq 1$.

THEOREM-4:

Let $A: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be A-concave function with $A: C \times C \rightarrow \mathbb{R}$ where C is convex vector space.

Let $A_0(0,0) = A(0,0,0)$

Assume that $f(0) = 0$

Then f is B_1 -subadditive function, where

$$\begin{aligned}
& B_1(x_1, x_2, x_3, x_4, x_5, x_6) - x_1 A_1\left(x_1, \sum_{i=1}^6 x_i\right) - x_2 A_1\left(x_2, \sum_{i=1}^6 x_i\right) \\
& - x_3 A_1\left(x_3, \sum_{i=1}^6 x_i\right) - x_4 A_1\left(x_4, \sum_{i=1}^6 x_i\right) - x_5 A_1\left(x_5, \sum_{i=1}^6 x_i\right) - x_6 A_1\left(x_6, \sum_{i=1}^6 x_i\right)
\end{aligned}$$

Proof: Recall that the function f is A-convex (A-concave) if

$$\frac{f(x) - f(z)}{x - z} \leq (\geq) \frac{f(y) - f(z)}{y - z} + A(x, y, z) \text{ with } x < z$$

$< y$.

By hypothesis f is A-Concave, then one can write,

$$\frac{f(x) - f(z)}{x - z} \leq \frac{f(y) - f(z)}{y - z} + A(x, y, z)$$

$$\Rightarrow (y-z)(f(x) - f(z)) \geq (x-z)(f(y) - f(z)) + (y-z)(x-z)A(x, y, z)$$

$$\begin{aligned} &\Rightarrow (y-z)f(x) - (y-z)f(z) \geq (x-z)f(y) - (x-z)f(z) + (y-z)(x-z)A(x,y,z) \\ &\Rightarrow (y-z)f(x) \geq (x-z)f(y) + (y-z-x+z)f(z) + (y-z)(x-z)A(x,y,z) \quad \frac{f(x_5)}{x_5} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_5, \sum_{i=1}^6 x_i\right), \\ &\Rightarrow (y-z)f(x) \geq (x-z)f(y) + (y-x)f(z) + (y-z)(x-z)A(x,y,z) \\ &\Rightarrow f(x) \geq \left(\frac{x-z}{y-z}\right)f(y) + \left(\frac{y-x}{y-z}\right)f(z) + (x-z)A(x,y,z) \quad \text{and } \frac{f(x_6)}{x_6} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_6, \sum_{i=1}^6 x_i\right). \end{aligned}$$

On simplification one can write,

$$\text{Take } \lambda = \frac{x-z}{y-z} \in (0,1)$$

$$\Rightarrow 1-\lambda = \frac{y-x}{y-z}$$

$$\text{Also } \lambda y + (1-\lambda)z = \frac{x-z}{y-z}y + \frac{y-x}{y-z}z = x$$

$$\text{Now } f(x) \geq \lambda f(y) + (1-\lambda)f(z) + (x-z)A(x,y,z)$$

By assumption $f(0) = 0$

$$\text{We have, } \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

Now from above, we observe that the function $\frac{f(x)}{x}$ is

A_1 -increasing.

So one can write

$$\frac{f(x_1)}{x} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_1, \sum_{i=1}^6 x_i\right),$$

$$\frac{f(x_2)}{x_2} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_2, \sum_{i=1}^6 x_i\right),$$

$$\frac{f(x_3)}{x} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_3, \sum_{i=1}^6 x_i\right),$$

$$\frac{f(x_4)}{x_4} \geq \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 (x_i)} + A_1\left(x_4, \sum_{i=1}^6 x_i\right),$$

$$\sum_{i=1}^6 f(x_i) \geq x_1 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} + x_2 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} + x_3 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i}$$

$$+ x_4 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} + x_5 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i} + x_6 \frac{f\left(\sum_{i=1}^6 x_i\right)}{\sum_{i=1}^6 x_i}$$

$$= x_1 A_1\left(x_1, \sum_{i=1}^6 x_i\right) + x_2 A_1\left(x_2, \sum_{i=1}^6 x_i\right) + x_3 A_1\left(x_3, \sum_{i=1}^6 x_i\right) + x_4 A_1\left(x_4, \sum_{i=1}^6 x_i\right)$$

$$+ x_5 A_1\left(x_5, \sum_{i=1}^6 x_i\right) + x_6 A_1\left(x_6, \sum_{i=1}^6 x_i\right)$$

$$= f\left(\sum_{i=1}^6 x_i\right) - B_1(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$= f\left(\sum_{i=1}^6 x_i\right) \leq \sum_{i=1}^6 f(x_i) + B_1(x_1, x_2, x_3, x_4, x_5, x_6)$$

$\Rightarrow f$ is B_1 -subadditive with given

$$B_1(x_1, x_2, x_3, x_4, x_5, x_6) = -x_1 A_1\left(x_1, \sum_{i=1}^6 x_i\right) - x_2 A_1\left(x_2, \sum_{i=1}^6 x_i\right) - x_3 A_1\left(x_3, \sum_{i=1}^6 x_i\right)$$

$$- x_4 A_1\left(x_4, \sum_{i=1}^6 x_i\right) - x_5 A_1\left(x_5, \sum_{i=1}^6 x_i\right) - x_6 A_1\left(x_6, \sum_{i=1}^6 x_i\right)$$

THEOREM -5:

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be convex and B -subadditive

Then function is C -increasing with $C : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$

Proof: Let $\lambda = \frac{5x}{5x+h} \in (0,1), h > 0$

$$\begin{aligned}
\therefore \lambda x + (1-\lambda)(6x+h) &= \frac{5x}{5x+h} \cdot x + \left(1 - \frac{5x}{5x+h}\right) (6x+h) \\
&= \frac{5x^2}{5x+h} + \frac{h}{5x+h} (6x+h) \\
&= \frac{5x^2}{5x+h} + \frac{6xh+h^2}{5x+h} \\
&= \frac{5x^2+6xh+h^2}{5x+h} \\
&= \frac{(x+h)(5x+h)}{5x+h} = x+h
\end{aligned}$$

Since f is B -subadditive, we have

$$\begin{aligned}
f(6x+h) &= f(x+x+x+x+x+h) \\
&\leq f(x)+f(x)+f(x)+f(x)+f(x)+f(x+h)+B(x,x,x,x,x,x+h) \\
&= 5f(x)+f(x+h)+B(x,x,x,x,x,x+h)
\end{aligned}$$

Since f is convex,

$$\begin{aligned}
f(x+h) &\leq \lambda f(x) + (1-\lambda)f(x+h) \\
&\leq \lambda f(x)(1-\lambda)[5f(x)+f(x+h)+B(x,x,x,x,x,x+h)] \\
&= \lambda f(x) + 5\lambda f(x) - 5\lambda f(x) + f(x+h) - \lambda f(x+h) + (1-\lambda)B(x,x,x,x,x,x+h) \\
&\Rightarrow \lambda f(x+h) \leq 5\lambda f(x) - 4\lambda f(x) + (1-\lambda)B(x,x,x,x,x,x+h) \\
&\Rightarrow \frac{5x}{5x+h} f(x+h) \leq 5f(x) - 4 \cdot \frac{5x}{5x+h} f(x) + \left(1 - \frac{5x}{5x+h}\right) B(x,x,x,x,x,x+h) \\
&\Rightarrow \frac{5x}{5x+h} f(x+h) \leq \frac{5(5x+h)f(x) - 20xf(x)}{5x+h} + \frac{5x}{5x+h} B(x,x,x,x,x,x+h) \\
&\Rightarrow \frac{5x}{5x+h} f(x+h) \leq \frac{5(5x+h)}{5x+h} f(x) + \frac{h}{5x+h} B(x,x,x,x,x,x+h) \\
&\Rightarrow 5xf(x+h) \leq 5(5x+h)f(x) + h \cdot B(x,x,x,x,x,x+h) \\
&\Rightarrow \frac{5xf(x+h)}{5x(x+h)} \leq \frac{f(x)}{x} + \frac{h \cdot B(x,x,x,x,x,x+h)}{5x(x+h)} \\
&\Rightarrow \frac{f(x+h)}{(x+h)} \leq \frac{f(x)}{x} + C(x,h) \text{ where} \\
C(x,h) &= \frac{h \cdot B(x,x,x,x,x,x+h)}{5x(x+h)}
\end{aligned}$$

Then function $\frac{f(x)}{x}$ is C -increasing function.

2. \wedge -INVEX FUNCTION:

Definition 2.1: Let V be a vector space and $S \subset V$ be η -invex subset of V where $\eta: V^6 \rightarrow V$ and $f: S \rightarrow \mathbb{R}_\infty$ where $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Then f is called η - \wedge -invex function if $f(p + \lambda \eta(p, q, r, s, t, u)) \leq \max\{f(p), f(q), f(r), f(s), f(t), f(u)\}$ for all $p, q, r, s, t, u \in S$ and $\lambda \in \wedge$.

DEFINITION 2.2

The set $D(f) = \{p \in S : f(p) < \infty\}$ is called effective domain of $f: S \rightarrow \mathbb{R}_+$.

DEFINITION 2.3

The point $p \in S$ with $f(p) = \infty$ is called a singular point of f . The set of all singular points of f is denoted by $S(f)$.

THEOREM 2.1

Let $f: V \rightarrow \mathbb{R}_\infty$ be η - \wedge -invex set and Let $K \subset D(f)$ be an open η -invex set.

Let us assume that $\eta: V^6 \rightarrow V$ be continuous and $f(p) > -\infty$ for all $p \in V$.

Then the function $f: K \rightarrow \mathbb{R}$ is η - C vazi-inve.

Proof: Let $p, q, r, s, t, u \in K$. Then there exist $a, b, c \in (0, 1)$ with

$$x = p + a\eta(p, q, r, s, t, u) \in K$$

$$y = q + b\eta(p, q, r, s, t, u) \in K$$

$$z = r + c\eta(p, q, r, s, t, u) \in K$$

Since we are in case of normed space, we can select sequences $(p_k), (q_k), (r_k), (s_k), (t_k)$ and (u_k) such that

$p_k \rightarrow p, q_k \rightarrow q, r_k \rightarrow r, s_k \rightarrow s, t_k \rightarrow t$ and $u_k \rightarrow u$ as $k \rightarrow \infty$

$$\Rightarrow f(p_k) \rightarrow \underline{f}(p), f(q_k) \rightarrow \underline{f}(q), f(r_k) \rightarrow \underline{f}(r)$$

$$f(s_k) \rightarrow \underline{f}(s), f(t_k) \rightarrow \underline{f}(t) \text{ and } f(u_k) \rightarrow \underline{f}(u) \text{ as } k \rightarrow \infty$$

Let $a_k, b_k, c_k \in \wedge$ be sequences such that $a_k \rightarrow a, b_k \rightarrow b$ and $c_k \rightarrow c$.

$$\text{Put } x_k = p_k + a_k \eta(p_k, q_k, r_k, s_k, t_k, u_k)$$

$$y_k = q_k + b_k \eta(p_k, q_k, r_k, s_k, t_k, u_k)$$

$$z_k = r_k + c_k \eta(p_k, q_k, r_k, s_k, t_k, u_k)$$

Then we have

$$x_k \rightarrow p + a\eta(p, q, r, s, t, u) = x$$

$$y_k \rightarrow q + b\eta(p, q, r, s, t, u) = y$$

$$z_k \rightarrow r + c\eta(p, q, r, s, t, u) = z$$

But $\underline{f}(x) \leq \liminf_{k \rightarrow \infty} \inf f(x_k)$ and

$$f(x_k) \leq \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

$$f(y) \leq \liminf_{k \rightarrow \infty} f(y_k) \text{ and } f(y_k) \leq \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

$$f(z) \leq \liminf_{k \rightarrow \infty} f(z_k) \text{ and } f(z_k) \leq \max\{f(p_k), f(q_k), f(r_k), f(s_k), f(t_k), f(u_k)\}$$

Taking the limit as $k \rightarrow \infty$ we get

$$\underline{f}(x) \leq \liminf_{k \rightarrow \infty} f(x_k) \leq \max\{\liminf_{k \rightarrow \infty} f(p_k), \liminf_{k \rightarrow \infty} f(q_k), \liminf_{k \rightarrow \infty} f(r_k), \liminf_{k \rightarrow \infty} f(s_k), \liminf_{k \rightarrow \infty} f(t_k), \liminf_{k \rightarrow \infty} f(u_k)\}$$

$$= \max\{\underline{f}(p), \underline{f}(q), \underline{f}(r), \underline{f}(s), \underline{f}(t), \underline{f}(u)\}$$

$$\underline{f}(y) \leq \liminf_{k \rightarrow \infty} f(y_k) \leq \max\{\liminf_{k \rightarrow \infty} f(p_k), \liminf_{k \rightarrow \infty} f(q_k), \liminf_{k \rightarrow \infty} f(r_k), \liminf_{k \rightarrow \infty} f(s_k), \liminf_{k \rightarrow \infty} f(t_k), \liminf_{k \rightarrow \infty} f(u_k)\}$$

$$= \max\{\underline{f}(p), \underline{f}(q), \underline{f}(r), \underline{f}(s), \underline{f}(t), \underline{f}(u)\}$$

$$\underline{f}(z) \leq \liminf_{k \rightarrow \infty} f(z_k) \leq \max\{\liminf_{k \rightarrow \infty} f(p_k), \liminf_{k \rightarrow \infty} f(q_k), \liminf_{k \rightarrow \infty} f(r_k), \liminf_{k \rightarrow \infty} f(s_k), \liminf_{k \rightarrow \infty} f(t_k), \liminf_{k \rightarrow \infty} f(u_k)\}$$

$$= \max\{\underline{f}(p), \underline{f}(q), \underline{f}(r), \underline{f}(s), \underline{f}(t), \underline{f}(u)\}$$

$\therefore f$ is η -Cvazi-invex.

Proposition 2.1: If $f : X \rightarrow \mathbb{R}_\infty$ is η -invex (or η -Cvazi-invex) then the set $D(f)$ is η -invex set (or η -Cvazi-invex set)

Proof: Let $p, q, r, s, t, u \in D(f)$

Then

$$f(p) < +\infty, f(q) < +\infty, f(r) < +\infty, f(s) < +\infty, f(t) < +\infty, f(u) < +\infty.$$

Then

$$f(p + \lambda\eta(p, q, r, s, t, u)) \leq \lambda f(q) + (1 - \lambda)f(r) < +\infty$$

(in η -invex case)

$$\Rightarrow f(p + \lambda\eta(p, q, r, s, t, u)) \leq \max\{f(p), f(q), f(r), f(s), f(t), f(u)\} < \infty$$

In any case $p + \lambda\eta(p, q, r, s, t, u) \in D(f)$

$$\Rightarrow D(f) \text{ is } \eta\text{-invex.}$$

THEOREM 2.2

Let V be a real banach space and η be a function such that for $M \subset V$ is $p, p_0, q, q_0, r, r_0 \in \text{int } M_0$ then there exists $\lambda \in [0, 1]$ and $u \in M$ such that

$$\left. \begin{aligned} p &= p_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \\ q &= q_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \\ r &= r_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0) \end{aligned} \right\} (*)$$

Let $f : X \rightarrow \mathbb{R}$ be η - \wedge -invex function and let

$p_0, q_0, r_0 \in \text{int } D(f)$ such that

$\bar{f}(p_0) \leq +\infty, \bar{f}(q_0) \leq +\infty, \bar{f}(r_0) \leq +\infty$ if η is non expansive related to the second argument then $\bar{f}(p) \leq +\infty, \bar{f}(q) \leq +\infty, \bar{f}(r) \leq +\infty$ for all $p, q, r \in \text{int } D(f)$.

Proof: Let $M = D(f)$ and let $p, p_0, q, q_0, r, r_0 \in D(f)$.

where $\bar{f}(p) = +\infty, \bar{f}(q) = +\infty, \bar{f}(r) = +\infty$

$$\bar{f}(p_0) < +\infty, \bar{f}(q_0) < +\infty, \bar{f}(r_0) < +\infty$$

By (*) and $u \in D(f)$, there exists $\lambda \in \wedge$ such that

$$p = p_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$

$$q = q_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$

$$r = r_0 + \lambda\eta(u, p_0, q_0, r_0, s_0, t_0)$$

Select now sequences $(p_k), (q_k),$ and (r_k) with

$$p_k \in D(f) - \{p\}, q_k \in D(f) - \{q\}, r_k \in D(f) - \{r\}$$

such that

$$p_k \rightarrow p, f(p_k) \rightarrow +\infty$$

$$q_k \rightarrow q, f(q_k) \rightarrow +\infty$$

$$r_k \rightarrow r, f(r_k) \rightarrow +\infty$$

There exist $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$f(p_k) > f(u) \text{ for } k \geq k_1$$

$$f(q_k) > f(u) \text{ for } k \geq k_2$$

$$f(r_k) > f(u) \text{ for } k \geq k_3$$

Let $k_0 = \max\{k_1, k_2, k_3\}$

So $f(p_k) > f(u), f(q_k) > f(u), f(r_k) > f(u)$ for all $k > k_0$

Let x_k, y_k and z_k be determined by the equations.

$$\left. \begin{aligned} p_k &= x_k + \lambda\eta(u, t, s, r, q, x_k) \\ q_k &= y_k + \lambda\eta(u, t, s, r, q, y_k) \\ r_k &= z_k + \lambda\eta(u, t, s, r, q, z_k) \end{aligned} \right\}$$

(1)

Equation (1) can be solved for all x_k, y_k and z_k (k fixed).

Since by setting $x_k = x, y_k = y$ and $z_k = z$ the map.

$$g(x) = p - \lambda\eta(u, t, s, r, q, x)$$

$$g_1(y) = q - \lambda\eta(u, t, s, r, q, y)$$

$$g_2(z) = r - \lambda \eta(u, t, s, r, q, z)$$

$g, g_1, g_2 : V \rightarrow V$ becomes a contradiction.

Indeed on has

$$\|g(x_1) - g(x_2)\| = \lambda \|\eta(u, t, s, r, q, x_1) - \eta(u, t, s, r, q, x_2)\| \leq \lambda < 1$$

$$\|g_1(y_1) - g_1(y_2)\| = \lambda \|\eta(u, t, s, r, q, y_1) - \eta(u, t, s, r, q, y_2)\| \leq \lambda < 1$$

also

$$\|g_2(z_1) - g_2(z_2)\| = \lambda \|\eta(u, t, s, r, q, z_1) - \eta(u, t, s, r, q, z_2)\| \leq \lambda < 1$$

η being non-expansive upon the third argument.

Now Banch's classical contradiction principle assures the existence of unique fixed point of operators g on g_1 and g_2 .

We shall now prove that

$$x_k \rightarrow p_0, y_k \rightarrow q_0 \text{ and } z_k \rightarrow r_0$$

$$\text{Now, } \|p_k - p\| = \|x_k - p + \lambda \eta(u, s, t, r, q, x_k)\|$$

$$= \|x_k - p_0 - \lambda \eta(u, s, t, r_0, q_0, p_0) + \lambda \eta(u, s, t, r, q, x_k)\|$$

$$\geq \|x_k - p_0\| - \lambda \|\eta(u, s, t, r_0, q_0, p_0) - \eta(u, s, t, r, q, x_k)\|$$

$$\geq \|x_k - p_0\| - \lambda \|x_k - p_0\|$$

$$\geq (1 - \lambda) \|x_k - p_0\|$$

$$\Rightarrow \|x_k - p_0\| \leq \frac{1}{1 - \lambda} \|p_k - p\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

Therefore $x_k \rightarrow p_0$ as $k \rightarrow \infty$

Similarly $y_k \rightarrow q_0$ as $k \rightarrow \infty$

and $z_k \rightarrow r_0$ as $k \rightarrow \infty$.

for $k > k_0$

$$f(u) < f(p_k) = \max\{f(x_k), f(q), f(r), f(s), f(t), f(u)\} = f(x_k)$$

$$f(u) < f(q_k) = \max\{f(y_k), f(q), f(r), f(s), f(t), f(u)\} = f(y_k)$$

$$f(u) < f(r_k) = \max\{f(z_k), f(q), f(r), f(s), f(t), f(u)\} = f(z_k)$$

Then $\bar{f}(p_0) \geq \lim_{k \rightarrow \infty} f(x_k) = +\infty$

$$\bar{f}(q_0) \geq \lim_{k \rightarrow \infty} f(y_k) = +\infty$$

$$\bar{f}(r_0) \geq \lim_{k \rightarrow \infty} f(z_k) = +\infty$$

which contradicts to

$$\bar{f}(p_0) < +\infty, \bar{f}(q_0) < +\infty \text{ and } \bar{f}(r_0) < +\infty$$

Remarks: If η has the non-expansivity property upon the arguments $p_0, q_0, r_0, s_0, t_0, u_0$ i.e.

$$\|\eta(p, q, r, s, t, u) - \eta(p_0, q_0, r_0, s_0, t_0, u_0)\|$$

$$\leq \|p - p_0\| + \|q - q_0\| + \|r - r_0\| + \|s - s_0\| + \|t - t_0\| + \|u - u_0\|,$$

then it is immediately seen that if $M \subset X$ as an invex set. Then $\text{int } M$ will also be invex.

REFERENCES

- [1] J. Sandor, Some Open Problems in the Theory of Functional Equations, Simpoziom on Applications of Fundamental Equations in Education, Science and Industry. (4th June, 1988), Odorhelue - Seculese, Romania.
- [2] J. Sandor, On the Principle of Condensation Singularities, Fourth National Conference on Mathematical Inequalities, Siblu, (30-31 Oct., 1992).
- [3] J. Sandor, Generalised Invexity and Its Application in Optimization Theory, First Joint Conference on Modern Applied Analysis, Sieni (Romania), (12-17 June, 1995)].
- [4] J. Sandor, On Certain Classes of Generalized Convex Functions with Application, I, Studia Univ, Babes-Bolyai, Math, 44, (1999), 73-84.
- [5] J. Sandor, On Cerntain Classes of Generalized Convex Functions with Application, II, Studia Univ, "Babes-Bolyai", Mathematica, XLVIII (1), (2002), 109-117.
- [6] R.B. Dash, D.K. Dalai and N. Mishra, A Note on a Class Generalised Convex Functions with some Applications, International Review of Pure and Applied Mathematical (July - December) 2009, Volume 5, No. 2, PP. 243-253.
- [7] B.C. Dash, and D.K. Dalai On a Class of Generalised Convex Function With Some Applications, American Journal of Mathematics and Mathematical Sciences, 4(1), January-June 2015. ISSN : 2278-0874. pp, 13-20.