

# New Properties of Optimal $\delta\theta$ -Open and $\delta\theta$ -Closed Sets in Topological Spaces

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## ABSTRACT

In this work, some conditions for  $\delta\theta$ -disconnectedness of a topological space in terms of maximal and minimal  $\delta\theta$ -open sets and also some similar results in terms of maximal and minimal  $\delta\theta$ -closed sets and also interrelationships between them are investigated. Generally, we find that if a space has a set which is both maximal and minimal  $\delta\theta$ -open, then either this set is the only non-trivial  $\delta\theta$ -open set in the space or the space is  $\delta\theta$ -disconnected. We also obtain a result concerning a minimal  $\delta\theta$ -open set on a subspace.

**Keywords :** Maximal  $\delta\theta$ -open set, minimal  $\delta\theta$ -open set, maximal  $\delta\theta$ -closed set, minimal  $\delta\theta$ -closed set,  $\delta\theta$ -disconnected.

## I. INTRODUCTION

In 1970, Norman Levine [5] introduced the notion of generalized closed sets and its dual open sets in topological spaces. After him many authors concentrated in this directions and defined various types of generalized closed sets in that spaces. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most wellknown notions and also an inspiration source is the notion of  $\delta$ -open and  $\theta$ -open sets introduced by N. V. Velicko [3] in 1968. Since the collection of  $\theta$ -open sets in a topological space  $(X, \tau)$  forms a topology  $\tau_\theta$  on  $X$ , then the union of two  $\theta$ -open sets is of course  $\theta$ -open. Moreover,  $\tau = \tau_\theta$  if and only if  $(X, \tau)$  is regular. F. Nakaoka and N. Oda in [1] and [2] introduced the notion of maximal open sets and minimal closed sets. Thereafter, J. Cao, M. Ganster, I. Reilly and M. Steiner in [4] introduced  $\delta\theta$ -open and  $\delta\theta$ -closed sets. In this paper, along with some other properties, we study a topological space in which a set may have double nature such as a set which is both maximal  $\delta\theta$ -open and minimal  $\delta\theta$ -open e.g. Theorem

2.9 and we show that the space may be  $\delta\theta$ -disconnected.

## II. PRELIMINARY NOTE

For a subset  $A$  of a topological space  $(X, \tau)$ ,  $\delta\theta\text{-Cl}(A)$  denotes the  $\delta\theta$ -closure of  $A$  with respect to the topological space  $(X, \tau)$ . Sometimes the topological space  $(X, \tau)$  is simply denoted by  $X$ . By a proper open set of a topological space  $X$ , we mean an open set  $G \neq \varnothing, X$  and by a proper closed set, we mean a closed set  $E \neq \varnothing, X$ . For a topological space  $(X, \tau)$  and  $A \subset X$ , we write  $(A, \tau_A)$  to denote the subspace on  $A$  of  $(X, \tau)$ . Throughout the work ordered pairs  $(X, \tau)$  and  $(Y, \sigma)$  (or  $X$  and  $Y$ ) will denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let  $A$  be a subset of  $X$ . We denote the interior and the closure of a set  $A$  by  $\text{Int}(A)$  and  $\text{Cl}(A)$ , respectively. Let us recall the following definitions which are useful in the sequel:

**Definition 2.1 [3].** Let  $A$  be a subset of a space  $X$ . A point  $x \in X$  is called a  $\theta$ -cluster point of  $A$  if  $A \cap \text{Cl}(U) \neq \varnothing$ , for every open set  $U$  of  $X$  containing  $x$ . The set of all  $\theta$ -cluster points of  $A$  is called the  $\theta$ -closure of  $A$ , denoted by  $\text{Cl}_\theta(A)$ .

**Definition 2.2 [3].** A subset  $A$  of  $X$  is called  $\theta$ -closed if  $A = Cl_\theta(A)$ . The complement of a  $\theta$ -closed set is called  $\theta$ -open set in  $X$ .

We denote the collection of all  $\theta$ -open (respectively,  $\theta$ -closed) sets by  $\theta\text{-}O(X, \tau)$  (respectively,  $\theta\text{-}C(X, \tau)$ ).

**Definition 2.3 [3].** Let  $A$  be a subset of a space  $X$ . A point  $x \in X$  is called a  $\delta$ -cluster point of  $A$  if  $A \cap U \neq \emptyset$ , for every regular open set  $U$  of  $X$  containing  $x$ . The set of all  $\delta$ -cluster points of  $A$  is called the  $\delta$ -closure of  $A$ , denoted by  $Cl_\delta(A)$ .

**Definition 2.4 [3].** A subset  $A$  of  $X$  is called  $\delta$ -closed if  $A = Cl_\delta(A)$ . The complement of a  $\delta$ -closed set is called  $\delta$ -open set in  $X$ .

We denote the collection of all  $\delta$ -open (respectively,  $\delta$ -closed) sets by  $\delta\text{-}O(X, \tau)$  (respectively,  $\delta\text{-}C(X, \tau)$ ).

It is very well known that the families of all  $\delta$ -open (resp.  $\theta$ -open) subsets of  $(X, \tau)$  are topologies on  $X$  which we shall denote by  $\tau_\delta$  (resp.  $\tau_\theta$ ). From the definitions it follows immediately that  $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ . The space  $(X, \tau_\delta)$  is also called the semi-regularization of  $(X, \tau)$ . A space  $(X, \tau)$  is said to be semi-regular if  $\tau_\delta = \tau$  and  $(X, \tau)$  is regular iff  $\tau_\theta = \tau$ . It is easily seen that one always has,  $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A) \subseteq \bar{A}^\theta$ , where,  $\bar{A}^\theta$  denotes the closure of  $A$  with respect to  $(X, \tau_\theta)$ .

**Definition 2.5 [1].** A proper nonempty open set  $U$  of  $X$  is said to be a maximal open set if any open set which contains  $U$  is either  $X$  or  $U$ .

**Definition 2.6 [2].** A proper nonempty closed set  $V$  of  $X$  is said to be a minimal closed set if any closed set contained in  $V$  is either  $\emptyset$  or  $V$ .

The family of all maximal open (resp. minimal closed) sets will be denoted by  $M_aO(X)$  (resp.  $M_iC(X)$ ). We define  $M_aO(X, x) = \{U : x \in U \in M_aO(X)\}$  and  $M_iC(X, x) = \{V : x \in V \in M_iC(X)\}$ .

**Definition 2.7[4].** A subset  $A$  of a topological space  $(X, \tau)$ , is called  $\delta\theta$ -closed [2] if  $Cl_\delta(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is  $\theta$ -open in  $X$ . The complement of  $\delta\theta$ -closed is  $\delta\theta$ -open set in  $X$ .

### III. MAXIMAL AND MINIMAL $\delta\theta$ -OPEN SETS

In this paper, we obtain some results on  $\delta\theta$ -disconnectedness of a topological space and hence we recall the following definition of  $\delta\theta$ -disconnectedness and some more results.

**Definition 2.1.** A topological space  $X$  is  $\delta\theta$ -disconnected if there exist disjoint nonempty  $\delta\theta$ -open sets  $G$  and  $H$  such that  $G \cup H = X$ .

**Definition 2.2.** A proper nonempty  $\delta\theta$ -open subset  $U$  of  $X$  is said to be a maximal  $\delta\theta$ -open set if any  $\delta\theta$ -open set which contains  $U$  is  $X$  or  $U$ .

**Theorem 2.3.** In any topological space  $(X, \tau)$ , if  $A$  be a maximal  $\delta\theta$ -open set and  $B$  be any  $\delta\theta$ -open set of  $(X, \tau)$ . Then either  $A \cup B = X$  or  $B \subset A$ .

**Proof.** Let  $A$  be a maximal  $\delta\theta$ -open set and  $B$  be a  $\delta\theta$ -open set of  $(X, \tau)$ . If  $A \cup B = X$ , then we are done. But if  $A \cup B \neq X$ , then we have to prove that  $B \subset A$ . Now  $A \cup B \neq X$  means  $B \subset A \cup B$  and  $A \subset A \cup B$ . Therefore, we have,  $A \subset A \cup B$  and  $A$  is maximal  $\delta\theta$ -open, then by definition,  $A \cup B = X$  or  $A \cup B = A$  but  $A \cup B \neq X$ , then  $A \cup B = A$  which implies  $B \subset A$ .

**Definition 2.4.** A nonempty  $\delta\theta$ -open subset  $U$  of  $X$  is said to be a minimal  $\delta\theta$ -open set if any  $\delta\theta$ -open set which is contained in  $U$  is  $U$  or  $\emptyset$ .

**Theorem 2.5.** If  $U$  is a minimal  $\delta\theta$ -open set and  $W$  is an  $\delta\theta$ -open set, then either  $U \cap W = \emptyset$  or  $U \subset W$ .

**Proof:** Easy and follows from definition.

**Theorem 2.6.** In any topological space  $(X, \tau)$ ,

(a) Let  $A$  be a maximal  $\delta\theta$ -open set of  $(X, \tau)$  and  $x$  an element of  $X \setminus A$ . Then for any  $\delta\theta$ -open set  $B$  containing  $x$ ,  $X \setminus A \subset B$ .

(b) Let  $A$  be a maximal  $\delta\theta$ -open set of  $(X, \tau)$ . Then, either of the following (i) and (ii) holds.

(i) For each  $x \in X \setminus A$  and each  $\delta\theta$ -open set  $B$  containing  $x$ ,  $B = X$ .

(ii) There exists a  $\delta\theta$ -open set  $B$  such that  $X \setminus A \subset B$  and  $B \subset X$ .

(c) Let  $A$  be a maximal  $\delta\theta$ -open set of  $(X, \tau)$ . Then, either of the following (i) and (ii) holds.

(i) For each  $x \in X \setminus A$  and each  $\delta\theta$ -open set  $B$  containing  $x$ , we have  $X \setminus A \subset B$ .

(ii) There exists a  $\delta\theta$ -open set  $B$  such that  $X \setminus A = B \neq X$ .

**Proof. (1)** Since  $x \in X \setminus A$ , we have  $B \not\subset A$  for any  $\delta\theta$ -open set  $B$  containing  $x$ . Then,  $A \cup B = X$  by Theorem 2.3. Therefore,  $X \setminus A \subset B$ .

(2) If (i) does not hold, then there exists an element  $x$  of  $X \setminus A$  and a  $\delta\theta$ -open set  $B$  containing  $x$  such that  $B \subset X$ . By (1), we have,  $X \setminus A \subset B$ .

(3) If (ii) does not hold, then, by (1), we have  $X \setminus A \subset B$  for each  $x \in X \setminus A$  and each  $\delta\theta$ -open set  $B$  containing  $x$ . Hence, we have  $X \setminus A \subset B$ .

**Theorem 2.7.** If  $G$  is a maximal  $\delta\theta$ -open set and  $H$  is a minimal  $\delta\theta$ -open set of a topological space  $X$ , then either  $H \subset G$  or the space is  $\delta\theta$ -disconnected.

**Proof.** Using the  $\delta\theta$ -maximality of  $G$  by Theorem 2.3, we get either  $G \cup H = X$  or  $H \subset G$ . Using the  $\delta\theta$ -minimality of  $H$  by Theorem 2.5, we get  $G \cap H = \varnothing$  or  $H \subset G$ . The case  $G \cup H = X$ ,  $H \subset G$  implies  $G = X$  and the case  $H \subset G$ ,  $G \cap H = \dots$  implies  $H = \varnothing$ . So only feasible cases are  $G \cup H = X$ ,  $G \cap H = \varnothing$  and  $H \subset G$ . If  $G \cup H = X$ ,  $G \cap H = \varnothing$ , then the space is  $\delta\theta$ -disconnected.

**Remark 2.8.**  $G \cup H = X$ ,  $G \cap H = \varnothing$  imply  $G = X - H$ . In Theorem 2.7, if  $H \not\subset G$ , then  $G$  and  $H$  both are also  $\delta\theta$ -closed. Theorem 2.7 may be stated as follows: If  $G$  is a maximal  $\delta\theta$ -open set and  $H$  is a minimal  $\delta\theta$ -open set of a topological space  $X$ , then either  $H \subset G$  or  $G = X - H$ .

**Theorem 2.9.** If a topological space  $X$  has a set which is both maximal and minimal  $\delta\theta$ -open sets, then either this set is the only nontrivial  $\delta\theta$ -open set in the space or the space is  $\delta\theta$ -disconnected.

**Proof.** Let  $G$  be both maximal and minimal  $\delta\theta$ -open, and  $H$  be any  $\delta\theta$ -open set. Then we get  $G \subset G \cup H$ . By the  $\delta\theta$ -maximality of  $G$ , we have the following two cases.

**Case I:**  $G = G \cup H$ . Then we get  $H \subset G$ . Since  $G$  is minimal  $\delta\theta$ -open set, we have,  $H = \varnothing$  or  $H = G$ .

**Case II:**  $G \cup H = X$ . Considering  $G$  as a minimal  $\delta\theta$ -open set, we get by Theorem 2.5,  $G \cap H = \varnothing$  or  $G \subset H$ . Since  $G$  is maximal  $\delta\theta$ -open,  $G \subset H$  implies  $G = H$  or  $H = X$ .

Considering all cases, we get  $G = H$  or  $G \cup H = X$  and  $G \cap H = \varnothing$ . If  $G \cup H = X$  and  $G \cap H = \varnothing$ , then the space is  $\delta\theta$ -disconnected.

**Remark 2.10:** It is trivial that if a topological space  $X$  has only one proper  $\delta\theta$ -open set, then that set is both maximal and minimal  $\delta\theta$ -open. If there are only two proper  $\delta\theta$ -open sets in a space and the  $\delta\theta$ -open sets are disjoint, then both are maximal and minimal  $\delta\theta$ -open sets. If  $G$  and  $H$  are only two proper  $\delta\theta$ -open sets in a topological space such that  $G \subset H$ , then  $G$  is a minimal  $\delta\theta$ -open set and  $H$  is a maximal  $\delta\theta$ -open set in the space. However, there may not exist a set which is both maximal and minimal  $\delta\theta$ -open in a  $\delta\theta$ -disconnected space which can be seen from Example 2.13.

**Corollary 2.11.** If  $G$  is both maximal and minimal  $\delta\theta$ -open, and  $E$  is a  $\delta\theta$ -closed set in a topological space  $X$ , then either  $G = X - E$  or  $G = E$ .

**Proof.** Given  $G$  is both maximal and minimal  $\delta\theta$ -open, and  $E$  is a  $\delta\theta$ -closed set. So  $X - E$  is an  $\delta\theta$ -open set. Proceeding like the proof of Theorem 2.9, we get,  $G = X - E$  or  $G \cup (X - E) = X$  and  $G \cap (X - E) = \varnothing$ .  $G \cup (X - E) = X$  and  $G \cap (X - E) = \varnothing$  imply  $G = E$ .

**Corollary 2.12.** If  $G$  is both maximal and minimal  $\delta\theta$ -open in a topological space  $X$ , then either  $G$  is the only proper  $\delta\theta$ -open set in the space or proper  $\delta\theta$ -open sets of the space are  $G$  and  $X - G$  only.

**Proof.** Let  $H$  be any proper  $\delta\theta$ -open set of the space. Proceeding like the proof of Theorem 2.8, we get  $G = H$  or  $G \cup H = X$  and  $G \cap H = \varnothing$ ,  $G \cup H = X$  and  $G \cap H = \varnothing$  imply  $H = X - G$ .

**Example 2.13.** Let us consider the topological space  $(X, \tau)$  such that  $X = \{a, b, c, d\}$  and  $\tau = \{X, \varnothing, \{c\}, \{c, d\}, \{a, b\}, \{a, b, c\}\}$ .

We find that  $\delta O(X, \tau) = \{X, \varnothing, \{c, d\}, \{a, b\}\}$ ;

$\delta\theta-C(X, \tau) = \{X, \varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{c, d\}, \{a, b\}\}$ .

$\delta\theta-O(X, \tau) = \{X, \varnothing, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$ .

Here, the topological space  $(X, \tau)$  is  $\delta\theta$ -disconnected since there exists two  $\delta\theta$ -open sets  $\{a, b\}$  and  $\{c, d\}$  such that  $X = \{a, b\} \cup \{c, d\}$  with  $\{a, b\} \cap \{c, d\} = \varnothing$ . But the space has no such  $\delta\theta$ -open set which is both maximal  $\delta\theta$ -open and minimal  $\delta\theta$ -open set.

**Theorem 2.14.** If  $A$  and  $B$  are two different maximal  $\delta\theta$ -open sets in a topological space  $X$  with  $A \cap B$  is a minimal  $\delta\theta$ -closed set, then  $X$  is  $\delta\theta$ -disconnected.

**Proof.** Since  $A$  and  $B$  are maximal  $\delta\theta$ -open set, we have  $A \cup B = X$ . We put  $G = A - A \cap B$ ,  $H = B$  or  $G = A$ ,  $H = B - A \cap B$ . We observe that  $G$  and  $H$  are disjoint  $\delta\theta$ -open sets with  $G \cup H = X$ . So,  $X$  is  $\delta\theta$ -disconnected.

**Theorem 2.15.** If  $U$  is a maximal  $\delta\theta$ -open set, then either  $\delta\theta-CI(U) = X$  or  $\delta\theta-CI(U) = U$ .

**Proof:** Since,  $U$  is a maximal  $\delta\theta$ -open set, then by Theorem 2.7, following two cases occur:

**Case I:** For each  $x \in X \setminus U$  and each  $\delta\theta$ -open set  $W$  containing  $x$ , we have,  $X \setminus U \subset W$ . Let  $x$  be any element of  $X \setminus U$  and  $W$  be any  $\delta\theta$ -open set containing  $x$ . Since  $X \setminus U \neq W$ , we have  $W \cap U \neq \varnothing$  for any  $\delta\theta$ -open set  $W$  containing  $x$ . Thus,  $X \setminus U \subset \delta\theta-CI(U)$ . Since  $X = X \cup (X \setminus U) \subset U \cup \delta\theta-CI(U) = \delta\theta-CI(U) \subset X$ , we have,  $\delta\theta-CI(U) = X$ .

**Case II:** There is an  $\delta\theta$ -open set  $W$  such that  $X \setminus U = W \neq X$ , Since  $X \setminus U = W$  is an  $\delta\theta$ -open set,  $U$  is an  $\delta\theta$ -closed set. Therefore,  $U = \delta\theta-CI(U)$ .

**Definition 2.16.** A nonempty  $e$ -open subset  $U$  of  $X$  is said to be  $e$ -dense set iff  $e-CI(U) = X$ .

**Theorem 2.17.** If there exists a maximal  $\delta\theta$ -open set which is not  $\delta\theta$ -dense in a topological space, then the top. space is  $\delta\theta$ -disconnected.

**Proof.** Suppose that  $U$  be a maximal  $\delta\theta$ -open set which is not  $\delta\theta$ -dense in  $X$ . By Theorem 2.15,  $U = \delta\theta\text{-Cl}(U)$ . If we take,  $G = U$  and  $H = X - \delta\theta\text{-Cl}(U)$ , then  $(G, H)$  is a  $\delta\theta$ -separation for  $X$ . Thus, the top. space is  $\delta\theta$ -disconnected.

#### IV. MAXIMAL AND MINIMAL $\delta\theta$ -CLOSED SETS

In this section, similar results in terms of maximal and minimal  $\delta\theta$ -closed sets along with interrelationships between them are investigated.

**Definition 3.1.** A proper nonempty  $\delta\theta$ -closed subset  $F$  of  $X$  is said to be a maximal  $\delta\theta$ -closed set if any  $\delta\theta$ -closed set which contains  $F$  is  $X$  or  $F$ .

**Theorem 3.2.** If  $E$  is a maximal  $\delta\theta$ -closed set and  $F$  is any  $\delta\theta$ -closed set, then either  $E \cup F = X$  or  $F \subset E$ .

**Definition 3.3.** A proper nonempty  $\delta\theta$ -closed subset  $E$  of  $X$  is said to be a minimal  $\delta\theta$ -closed set if any  $\delta\theta$ -closed set which is contained in  $E$  is  $E$  or  $\varnothing$ .

**Theorem 3.4.** In any topological space  $(X, \tau)$ , if  $F$  be a minimal  $\delta\theta$ -closed set and  $G$  be a  $\delta\theta$ -closed set of  $(X, \tau)$ . Then either  $F \cap G = \varnothing$  or  $F \subset G$ .

**Proof.** Let  $F$  be a minimal  $\delta\theta$ -closed set and  $G$  be a  $\delta\theta$ -closed set of  $(X, \tau)$ . If  $F \cap G = \varnothing$ , then there is nothing to prove. But if  $F \cap G \neq \varnothing$ , then we have to prove that  $F \subset G$ . Now if  $F \cap G \neq \varnothing$ , then  $F \cap G \subset F$  and  $F \cap G \subset G$ . Since  $F \cap G \subset F$  and given that  $F$  is minimal  $\delta\theta$ -closed, then by definition  $F \cap G = F$  or  $F \cap G = \varnothing$ . But  $F \cap G \neq \varnothing$ , then  $F \cap G = F$  which shows that  $F \subset G$ .

By Theorem 2.5 and Theorem 3.4, we note that if  $G$  is both minimal  $\delta\theta$ -open and minimal  $\delta\theta$ -closed, and  $E$  is  $\delta\theta$ -clopen, then either  $G \subset E$  or  $G \cap E = \varnothing$ . Analogous to Theorem 2.7, Theorem 2.9, Corollary 2.11 and Corollary 2.12, we have Theorem 3.5, Theorem 3.6, Corollary 3.7 and Corollary 3.8 respectively. Since proofs of theorems and corollaries are similar to the proof of corresponding theorems and corollaries already establish, we omit proofs.

**Theorem 3.5.** If  $E$  is a maximal  $\delta\theta$ -closed set and  $F$  is a minimal  $\delta\theta$ -closed set of a topological space  $X$ , then either  $F \subset E$  or  $E \cup F = X$ ,  $E \cap F = \varnothing$ .

**Theorem 3.6.** If a topological space  $X$  has a set  $E$  which is both maximal and minimal  $\delta\theta$ -closed, then either of the following is true.

(i)  $E$  is the only proper  $\delta\theta$ -closed set in the space.

(ii) If there exists another proper  $\delta\theta$ -closed set  $F$ , then  $E \cup F = X$  and  $E \cap F = \varnothing$ .

**Corollary 3.7.** If  $E$  is both maximal and minimal  $\delta\theta$ -closed, and  $G$  is an  $\delta\theta$ -open set in a topological space  $X$ , then either  $E = X - G$  or  $E = G$ .

**Corollary 3.8.** If  $E$  is both maximal and minimal  $\delta\theta$ -closed in a topological space  $X$ , then either  $E$  is the only proper  $\delta\theta$ -closed set in the space or proper  $\delta\theta$ -closed sets of the space are  $E$  and  $X - E$  only.

**Remark 3.9:** It is trivial that if a topological space  $X$  has only one proper  $\delta\theta$ -closed set, then that set is both maximal and minimal  $\delta\theta$ -closed. If there are only two proper  $\delta\theta$ -closed sets in a space and the  $\delta\theta$ -closed sets are disjoint, then both are maximal and minimal  $\delta\theta$ -closed. If  $E$  and  $F$  are only two proper  $\delta\theta$ -closed sets in a topological space such that  $F \subset E$ , then  $F$  is a minimal  $\delta\theta$ -closed and  $E$  is a maximal  $\delta\theta$ -closed set in the space.

**Example 3.10.** In the topological space of Example 2.13, there exist disjoint proper  $\delta\theta$ -closed sets  $\{a, b\}$  and  $\{c, d\}$  such that  $X = \{a, b\} \cup \{c, d\}$ . But there is no proper  $\delta\theta$ -closed set in the space which is both maximal and minimal  $\delta\theta$ -closed set. Thus, we conclude that there may exist closed sets  $E, F$  in  $X$  such that  $E \cup F = X$  and  $E \cap F = \varnothing$  but there may not exist a set which is both maximal and minimal  $\delta\theta$ -closed.

**Theorem 3.11.** If  $G$  is both maximal  $\delta\theta$ -open and minimal  $\delta\theta$ -closed,  $H$  is open and  $E$  is  $\delta\theta$ -closed, then either of the following is true.

(i)  $H \subset G \subset E$ .

(ii)  $H \subset G$  and  $G \cap E = \varnothing$ .

(iii)  $G \cup H = X$  and  $G \subset E$ .

(iv)  $G \cup H = X$ ,  $G \cap E = \varnothing$ .

**Proof.** Considering  $G$  as a maximal  $\delta\theta$ -open set, By Theorem 2.3 we get  $H \subset G$  or  $G \cup H = X$ . Considering  $G$  as a minimal  $\delta\theta$ -closed set, by Theorem 3.4,  $G \subset E$  or  $G \cap E = \varnothing$ .  $H \subset G$  and  $G \subset E$  imply  $H \subset G \subset E$ . The remaining probable combinations are  $H \subset G$ ,  $G \cap E = \varnothing$ ;  $G \cup H = X$ ,  $G \subset E$  and  $G \cup H = X$ ,  $G \cap E = \varnothing$ .

**Corollary 3.12.** If  $G$  is both maximal  $\delta\theta$ -open and minimal  $\delta\theta$ -closed, then  $G$  and  $X - G$  are only proper  $\delta\theta$ -clopen sets in the space.

**Proof.** Let  $E$  be  $\delta\theta$ -clopen in  $X$ . Putting  $H = E$  in Theorem 3.11, we get  $G = E$  or  $G = X - E$ .

**Theorem 3.13.** If  $G$  is both maximal  $\delta\theta$ -open and maximal  $\delta\theta$ -closed, and  $E$  is  $\delta\theta$ -clopen, then either  $E \subset G$  or  $G \cup E = X$ .

**Proof.** Similar to the proof of Theorem 3.11. Corresponding to Theorem 3.11 and Corollary 3.12, we have Theorem 3.14 and Corollary 3.15 respectively. The proofs of the theorem and the corollary are omitted as they are similar to proofs already established.

**Theorem 3.14.** If  $G$  is both minimal  $\delta\theta$ -open and maximal  $\delta\theta$ -closed,  $H$  is  $\delta\theta$ -open and  $E$  is  $\delta\theta$ -closed, then either of the following is true.

- (i)  $E \subset G \subset H$ .
- (ii)  $G \subset H, G \cup E = X$ .
- (iii)  $G \cap H = \emptyset, E \subset G$ .
- (iv)  $G \cup E = X, G \cap H = \emptyset$ .

**Corollary 3.15.** If  $G$  is both minimal  $\delta\theta$ -open and maximal  $\delta\theta$ -closed, then  $G$  and  $X - G$  are only proper  $\delta\theta$ -clopen sets in the space.

**Theorem 3.16.** Let  $A, G$  be  $\delta\theta$ -open sets in  $X$  such that  $A \cap G \neq \emptyset$ . Then,  $A \cap G$  is a minimal  $\delta\theta$ -open set in  $(A, \tau_A)$  if  $G$  is a minimal  $\delta\theta$ -open set in  $(X, \tau)$ .

**Proof.** If  $A \cap G$  is not a minimal  $\delta\theta$ -open set in  $(A, \tau_A)$ , there exists an  $\delta\theta$ -open set  $U \neq \emptyset$  in  $(A, \tau_A)$  such that  $U \subsetneq A \cap G$ . Since  $G$  is a minimal  $\delta\theta$ -open set in  $X$  and  $A \cap G \neq \emptyset$ , we have by Theorem 2.5,  $G \subset A$  which implies  $A \cap G = G$ .  $A$  being  $\delta\theta$ -open in  $X$ ,  $U$  is also  $\delta\theta$ -open in  $X$ . So we get a set  $U$   $\delta\theta$ -open in  $X$  such that  $\emptyset \neq U \subsetneq G$  which is a contradiction to our assumption that  $G$  is a minimal  $\delta\theta$ -open set in  $X$ .

**Theorem 3.17.** Let  $U$  be a maximal  $\delta\theta$ -open set. Then  $\delta\theta\text{-Int}(X \setminus U) = X \setminus U$  or  $\delta\theta\text{-Int}(X \setminus U) = \emptyset$ .

**Proof.** By Theorem 2.6, we have following cases (1)  $\delta\theta\text{-Int}(X \setminus U) = \emptyset$  or (2)  $\delta\theta\text{-Int}(X \setminus U) = X \setminus U$ .

**Theorem 3.18.** Let  $U$  be a maximal  $\delta\theta$ -open set and  $S$  a nonempty subset of  $X \setminus U$ . Then  $\delta\theta\text{-Cl}(S) = X \setminus U$ .

**Proof.** Since  $\emptyset \neq S \subset X \setminus U$ , we have  $W \cap S \neq \emptyset$  for any element  $x$  of  $X \setminus U$  and any  $\delta\theta$ -open set  $W$  of  $x$  by Theorem 2.15. Then  $X \setminus U \subset \delta\theta\text{-Cl}(U)$ . Since  $X \setminus U$  is a  $\delta\theta$ -closed set and  $S \subset X \setminus U$ , we see that  $\delta\theta\text{-Cl}(S) \subset \delta\theta\text{-Cl}(X \setminus U) = X \setminus U$ . Therefore  $X \setminus U = \delta\theta\text{-Cl}(S)$ .

**Corollary 3.19.** Let  $U$  be a maximal  $\delta\theta$ -open set and  $M$  a subset of  $X$  with  $U \subset M$ . Then  $\delta\theta\text{-Cl}(M) = X$ .

**Proof.** Since  $U \subset M \subset X$ , there exists a nonempty subset  $S$  of  $X \setminus U$  such that  $M = U \cup S$ . Hence we have  $\delta\theta\text{-Cl}(M) = \delta\theta\text{-Cl}(U \cup S) = \delta\theta\text{-Cl}(U) \cup \delta\theta\text{-Cl}(S) \supset (X \setminus U) \cup U = X$  by Theorem 3.18. Therefore  $\delta\theta\text{-Cl}(M) = X$ .

**Theorem 3.20.** Let  $U$  be a maximal  $\delta\theta$ -open set and assume that the subset  $X \setminus U$  has two element at least. Then  $\delta\theta\text{-Cl}(X \setminus \{a\}) = X$  for any element of  $X \setminus U$ .

**Proof.** Since  $U \subset X \setminus \{a\}$  by our assumption, we have the result by Corollary 3.19.

**Theorem 3.21.** Let  $U$  be a maximal  $\delta\theta$ -open set, and  $N$  be a proper subset of  $X$  with  $U \subset N$ . Then,  $\delta\theta\text{-Int}(N) = U$ .

**Proof.** If  $N = U$ , then  $\delta\theta\text{-Int}(N) = \delta\theta\text{-Int}(U) = U$ . Otherwise,  $N \neq U$ , and hence  $U \subset N$ . It follows that  $U \subset \delta\theta\text{-Int}(N)$ . Since  $U$  is a maximal  $\delta\theta$ -open set, we also have,  $\delta\theta\text{-Int}(N) \subset U$ . Therefore,  $\delta\theta\text{-Int}(N) = U$ .

## V. CONCLUSION

In this work, some conditions for  $\delta\theta$ -disconnectedness of a topological space in terms of maximal and minimal  $\delta\theta$ -open sets and also some similar results in terms of maximal and minimal  $\delta\theta$ -closed sets along with interrelationships between them are investigated. Generally, we find that if a space has a set which is both maximal and minimal  $\delta\theta$ -open, then either this set is the only non trivial  $\delta\theta$ -open set in the space or the space is  $\delta\theta$ -disconnected. We also obtain a results concerning a minimal  $\delta\theta$ -open set on a subspace. Some example, counter examples and some of their fundamental results are also established. Hope that the findings in this paper will help researcher enhance and promote the further study on topological spaces to carry out a general framework for their applications in separation axioms, connectedness, compactness etc. and also in practical life.

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