# A Study on Magic Labeling Regular Graph 

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#### Abstract

Let $G(V, E)$ be a graph and $\lambda$ be a bisection from the set $V \cup E$ to the set of the first $|V|+|E|$ natural numbers. The weight of a vertex is the sum of its label and the labels of all adjacent edges. We say $\lambda$ is a vertex magic total (VMT) labeling of $G$ if the weight of each vertex is constant. We say $\lambda$ is an ( $s, d$ ) -vertex anti-magic total (VAT) labeling if the vertex weights form an arithmetic progression starting at s with difference $\mathrm{d} . \mathrm{J}$. MacDougall conjectured that any regular graph with the exception of K 2 and 2 K 3 has a VMT labeling. We give constructions of VAT labelings of any even-regular graphs and VMT labelings of certain regular graphs.


Keywords: Graph Labeling, Vertex Magic Total Labeling, Vertex Antimagic Total Labeling.

## I. INTRODUCTION

A labeling of a graph $G(V, E)$ is a mapping from the set of vertices, edges, or both vertices and edges to the set of labels. Based on the domain we distinguish vertex labelings, edge labelings and total labelings. In most applications the labels are positive (or nonnegative) integers, though in general real numbers could be used. Various labelings are obtained based on the requirements put on the mapping. Magic labelings were introduced by Sedlacek in 1963. In general for a magic-type labeling we require the sum of labels related to a vertex (a vertex magic labeling) or to an edge (an edge magic labeling) to be constant all over the graph. In an anti-magic labeling we require that all the sums (weights) are different. For many graphs this is not difficult to achieve, but not in general. The conjecture by Hartsfield and Ringel "All graphs except K2 are anti-magic" is still open.The concept of graph labeling was introduced by Rosa in 1967. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labelled graphs serve as useful models for broad range of applications such as coding theory, X-ray, crystallography, radar, astronomy, circuit design, communication networks and database management. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, anti-magic labeling, bi-magic labeling,
prime labeling, cordial labeling etc. have been considered in the literature. A graph $\mathrm{G}=\mathrm{G}(\mathrm{V}, \mathrm{E})$ with V vertices is said to admit prime labeling, if its vertices can be labeled with distinct positive integers, not exceeding V such that the labels of each pair of adjacent vertices are relatively prime.

## II. METHODS AND MATERIAL

## Basic Definition of Magic Labelling of Regular Graph

2.1. Definition: Let G be a graph with vertex set V and edge set E . We denote $\mathrm{v}=|\mathrm{V}|$ and $\mathrm{e}=|\mathrm{E}|$. Let $\lambda$ be a one-to-one mapping $\lambda: \mathrm{V} \cup \mathrm{E} \rightarrow\{1,2, \ldots, \mathrm{v}+\mathrm{e}\}$. The labeling $\lambda$ is called a vertex magic total (VMT) labeling of $G$ if there exists a constant $h$ such that for every vertex $x \in V$ is $w \lambda(x)=h$. The constant $h$ is the magic constant for $\lambda$. The labeling $\lambda$ is called a vertex anti-magic total labeling of $G$ if the vertex weights (1) are pairwise different for all vertices of G. If the weights form an arithmetic progression $\mathrm{s}, \mathrm{s}+\mathrm{d} \ldots \mathrm{s}+$ ( $\mathrm{v}-1$ )d, then $\lambda$ is called an ( $\mathrm{s}, \mathrm{d}$ ) -vertex anti-magic total (VAT) labeling of G .

A graph is a VMT graph if it admits a VMT labeling and similarly a VAT graph allows a VAT labeling. Examples of VMT and VAT labelings are given in Sections 3 and 4. We use the notation $\mathrm{v}=|\mathrm{V}|$ and $\mathrm{e}=$ $|\mathrm{E}|$ through the rest of this section.
2.2. Definition: A graph with $p$ vertices and edges is called total edge-magic if there is a bisection $\mathrm{f}: \mathrm{VUE} \rightarrow$ $\{1,2, \ldots \ldots .$. , $\mathrm{p}+\mathrm{q}\}$ such that there exists a constant ' s ' for any ( $u, v$ ) in E satisfying $f(u)+f(u . v)+f(v)=s$. The original concept of total edge-magic graph is due to Kotzig and Rosa.
2.3. Definition:A total edge magic graph is called a super edge magic graph if $\mathrm{f}:(\mathrm{V}(\mathrm{G})) \rightarrow\{1,2, \ldots \ldots \mathrm{p}\}$. The concept was introduced by Hikoe Enomoto et al., in 1998. A merge graph $\mathrm{G}_{1} * \mathrm{G}_{2}$ can be formed from two graphs $G_{1}$ and $G_{2}$ by merging a node of $G_{1}$ with a node of G2. As an example let us consider $\mathrm{T}_{3}$, a tree with three vertices and S 2 a star on three vertices then $\mathrm{T}_{3} *$ $S_{2}$ is formed as follows. Consider a vertex $b$ of $T_{3}$. Consider a vertex $\mathrm{v}_{1}$ of $\mathrm{S}_{2}$.
2.4. Definition:Let $G=(V, E)$ be a graph with a finite set of vertices $V(G)$ together with a set $E(G)$ of edges. Each edge e connects two vertices $\mathrm{x}, \mathrm{y} \in \mathrm{V}, \mathrm{e}=(\mathrm{x}, \mathrm{y})$. The cardinality of G is defined by $\mathrm{n}=|\mathrm{V}(\mathrm{G})|$, while m $=|\mathrm{E}(\mathrm{G})|$ defines the size of G . The edges and vertices of a graph are also said to be the graph elements. The $n c m$ graph is edge-magic for $\mathrm{m}=3$ and $\mathrm{n}=2,3$

Example:The ncm graphs have edge-magic labeling for $\mathrm{m}=3$ and $\mathrm{n}=2,3$ as shown below.



Figure
2.5. Definition: Graph theory is the study of mathematical structures called graphs. We define a graph as a pair ( $\mathrm{V}, \mathrm{E}$ ), where V is a nonempty set, and E is a set of unordered pairs of elements of $\mathrm{V} . \mathrm{V}$ is called the set of vertices of $G$, and $E$ is the set of edges. Two vertices a and b are adjacent provided $(\mathrm{a}, \mathrm{b}) \in \mathrm{E}$. If a pair of vertices is adjacent, the vertices are referred to as neighbors. We can represent a graph by representing the vertices as points and the edges as line segments connecting two vertices, where vertices $\mathrm{a}, \mathrm{b} \in$ V are connected by a line segment if and only if ( $\mathrm{a}, \mathrm{b}$ ) $\in E$. Figure 1 is an example of a graph with vertices $V$ $=\{\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}\}$ and edges $\mathrm{E}=\{(\mathrm{x}, \mathrm{w}),(\mathrm{z}, \mathrm{w}),(\mathrm{y}, \mathrm{z})\}$.

2.6. Definition:A walk of length k is a sequence of vertices $v_{0}, v_{1}, \ldots, v_{k}$, such that for all $i>0, v_{i}$ is adjacent to $v_{i-1}$. A connected graph is a graph such that for each pair of vertices $v_{1}$ and $v_{2}$ there exists a walk beginning at $v_{1}$ and ending at $v_{2}$. The graph in Figure 2 is not connected because there is no walk beginning at z and ending at w .

2.7.Definition: A cycle of length $\mathrm{k}>2$ is a walk such that each vertex is unique except that $\mathrm{v}_{0}=\mathrm{v}_{\mathrm{k}}$. From this point on, we will also assume that every graph discussed has at most one edge connecting each pair of vertices. That is, we assume that there are no 2 -cycles. A tree is a graph with no cycles. The girth of a graph is the length of its shortest cycle. Since a tree has no cycles, we define its girth as inf $\emptyset=\infty$.

2.1.Theorem:Let $\mathrm{G}:(\sigma, \mu)$ be a k - regular graph. Then $d_{G}(u)=k$, for all $u \in V$. Assume that $\mu$ is a constant function. Then to prove that G is edge regular.

Proof :Let $\mu=c$. By the definition of edge degree, $d_{G}(u v)=d_{G}(u)+d_{G}(v)-2 \mu(u v)$, for all uv $\in E .=k+$ $\mathrm{k}-2 \mathrm{c}$
$\Rightarrow \mathrm{d}_{\mathrm{G}}(\mathrm{uv})=\mathrm{k} 1$, for all $u v \in \mathrm{E}$, where $\mathrm{k} 1=\mathrm{k}+\mathrm{k}-2 \mathrm{c}$. Hence $G$ is edge regular. Conversely, assume that $G$ is edge regular. Then to prove that $\mu$ is a constant function. Let $d_{G}(u v)=k 1$, for all $u v \in E$.
By the definition of edge degree,
$d_{G}(u v)=d_{G}(u)+d_{G}(v)-2 \mu(u v)$, for all uv $\in E . k 1=$ $\mathrm{k}+\mathrm{k}-2 \mu$ (uv)
$\Rightarrow \mu(u v)=(2 k-k 1) / 2=c$, for all $u v \in E$, where $c=(2 k$ - k1)/2.

Hence $\mu$ is a constant function.
2.2.Theorem:For every $n \geq 5$, there exists a 4-regular ( $\mathrm{n}, 2 \mathrm{n}$ )-Bimagic graph of girth 3 with constants $4 \mathrm{n}-1$ and $5 \mathrm{n}-1$.
Proof: From the structure of 4-regular graph of girth $j$ where $\mathrm{j}=3$, we have n vertices and 2 n edges. To prove that for every $\mathrm{n} \geq 5$, there exists a 4-regular graph of girth 3 which admits bi-magic labeling with magic constants $4 \mathrm{n}-1$ and $5 \mathrm{n}-1$. i.e., we have to show that the induced function $f^{*}: V \rightarrow N$ is defined as $f^{*}\left(v_{i}\right)=\left\{\Sigma f\left(v_{i}\right.\right.$ $\left.v_{j}\right)=$ either $k_{1}$ or $k_{2}$, where $k_{1} \& k_{2}$ are constants and $v_{i} v_{j}$ $\in E\}$.
Define the map $f$ on $E$ as follows:
Let $\mathrm{f}: \mathrm{E} \rightarrow\{1,2,3 \ldots \ldots . .2 \mathrm{n}\}$ such that
(i) $f\left(v_{i}, v_{i+1}\right)=i, 1 \leq i \leq n-1$;
(ii) $\mathrm{f}\left(\mathrm{v}_{\mathrm{n}}, \mathrm{v}_{1}\right)=\mathrm{n}$;
(iii) $\mathrm{f}\left(\mathrm{v}_{\mathrm{i}+2}, \mathrm{v}_{\mathrm{i}}\right)=2 \mathrm{n}-\mathrm{i}-1,1 \leq \mathrm{i} \leq \mathrm{n}-2$;
(iv) $f\left(v_{1}, v_{n-1}\right)=2 n$
(v) $\mathrm{f}\left(\mathrm{v}_{2}, \mathrm{v}_{\mathrm{n}}\right)=2 \mathrm{n}-1$.
$(n-2)+1+1=2 n$. In order to get the labels on vertices define the induced map $f^{*}$ on $V$ as $f^{*}: V \rightarrow N$ is defined such that $f^{*}\left(v_{i}\right)=\left\{\Sigma f\left(v_{i} v_{j}\right) / v_{i} v_{j} \in E\right\}$. Now for every $\mathrm{v}_{\mathrm{i}} \in \mathrm{V}, \mathrm{f}^{*}\left(\mathrm{v}_{2}\right)=1+2+(2 \mathrm{n}-3)+(2 \mathrm{n}-1)$
[By (iii), (iv)]
$\mathrm{F} *\left(\mathrm{v}_{\mathrm{i}}\right)=[2 \mathrm{n}-(\mathrm{i}-2)-1]+(\mathrm{i}-1)+\mathrm{i}+(2 \mathrm{n}-\mathrm{i}-1)$ where $\mathrm{i}=$ 3 to $n-2$. [By (iii), (i), (iii)]
$\mathrm{f} *\left(\mathrm{v}_{1}\right)=1+(2 \mathrm{n}-2)+2 \mathrm{n}+\mathrm{n}[\mathrm{By}$ (iii), (iv), (ii) ]
$\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-1}\right)=2 \mathrm{n}+(\mathrm{n}-2)+(\mathrm{n}+2)+(\mathrm{n}-1)$ [By (iv), (i), (iii]
$\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}}\right)=\mathrm{n}+(2 \mathrm{n}-1)+(\mathrm{n}-1)+(\mathrm{n}+1)$ [ By (ii), (v), (i),
(iii) ]

Thus $\mathrm{f}^{*}\left(\mathrm{v}_{2}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{3}\right)=\ldots . .=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-2}\right)=4_{\mathrm{n}-1}$ and $\mathrm{f}^{*}\left(\mathrm{v}_{1}\right)=$ $\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-1}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}}\right)=5 \mathrm{n}-1$.

## Example:

4- Regular $(8,16)$ graph of girth 3 admits Bimagic labeling.


Hence the 4- Regular $(8,16)$ graph of girth 3 admits bi-magic labeling with magic constants 31 and 39.
2.3.Theorem : For every $n \geq 7$, there exists a 4-regular ( $\mathrm{n}, 2 \mathrm{n}$ ) Bi-magic graph of girth 4 with magic constants $4 n-2$ and $5 n-2$.

Proof:From the structure of 4-regular graph of girth $j$ where $j=4$, we have $n$ vertices and $2 n$ edges. To prove that, for $\mathrm{n} \geq 7$ with girth j where $\mathrm{j}=4$ there exists a 4regular graph which admits bi-magic labeling with magic constants $4 \mathrm{n}-2$ and $5 \mathrm{n}-2$. i.e., we have to show that the induced function.
$\mathrm{f}^{*}: V \rightarrow \mathrm{~V}$ is defined as $\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{i}}\right)=\left\{\Sigma \mathrm{f}\left(\mathrm{v}_{\mathrm{i}} \mathrm{v}_{\mathrm{j}}\right)=\right.$ either $\mathrm{k}_{1}$ or $\mathrm{k}_{2}$ where $\mathrm{k}_{1} \& \mathrm{k}_{2}$ are two constants and $\left.\mathrm{v}_{\mathrm{i}} \mathrm{v}_{\mathrm{j}} \in \mathrm{E}\right\}$.
Define the map $f$ on $E$ as follows:
Let $\mathrm{f}: \mathrm{E} \rightarrow\{1,2,3 \ldots \ldots . .2 \mathrm{n}\}$ such that
(i) $\mathrm{f}\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i}+1}\right)=\mathrm{i}, 1 \leq \mathrm{i} \leq \mathrm{n}-1$;
(ii) $f\left(v_{n}, v_{1}\right)=n$;
(iii) $f\left(v_{i+3}, v_{i}\right)=2 n-i-2,1 \leq i \leq n-3$;
(iv) $f\left(v_{1}, v_{n-2}\right)=2 n$
(v) $\mathrm{f}\left(\mathrm{v}_{2}, \mathrm{v}_{\mathrm{n}-1}\right)=2 \mathrm{n}-1$.
(vi) $f\left(v_{3}, v_{n}\right)=2 n-2$.

Thus, Number of labels assigned to edges $=(n-1)+1+$ $(n-3)+1+1+1=2 n$. In order to get the labels on vertices define the induced map $\mathrm{f}^{*}$ on V as $\mathrm{f}^{*}: \mathrm{V} \square \mathrm{N}$ is defined such that $f^{*}\left(v_{i}\right)=\left\{\Sigma f\left(v_{i} v_{j}\right) / v_{i} v_{j} \in E\right\}$.
Now for every $v_{i} \in V, f^{*}\left(v_{i}\right)=(i-1)+i+(2 n-(i-2)-$ 1) $+(2 n-i-2)$ where $i=2$ to $n-3$
[By (i), (iii)]
$\mathrm{F}^{*}\left(\mathrm{v}_{1}\right)=1+\mathrm{n}+2 \mathrm{n}+(2 \mathrm{n}-3)[\mathrm{By}$ (ii), (iv), (iii) $]$
$\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-2}\right)=\mathrm{n}-2+\mathrm{n}-3+2 \mathrm{n}+\mathrm{n}+3$. [By (i), (i), (iv), (i)]
$\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-1}\right)=(\mathrm{n}-1)+(\mathrm{n}-2)+(2 \mathrm{n}-1)+(\mathrm{n}+2)[$ By (i), (v),
(iii)]
$\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}}\right)=\mathrm{n}+(2 \mathrm{n}-2)+(\mathrm{n}-1)+(\mathrm{n}+1)[$ By (ii), (vi), (i), (iii)]

Thus $\mathrm{f}^{*}\left(\mathrm{v}_{2}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{3}\right)=\ldots \ldots \ldots . .=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-3}\right)=4 \mathrm{n}-2$ and $\mathrm{f}^{*}$ $\left(\mathrm{v}_{1}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-2}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}-1}\right)=\mathrm{f}^{*}\left(\mathrm{v}_{\mathrm{n}}\right)=5 \mathrm{n}-2$.

## Example:

4-Regular $(8,16)$ graph of girth 4 admits Bi-magic labelling


## Hence, the 4-Regular $(8,16)$ graph of girth 4 admits bi-magic labeling with magic constants 30 and 38.

### 2.4.Theorem: Petersen Theorem:

Let $G$ be a $2 r$-regular graph. Then there exists a 2 factor in G .
Notice that after removing edges of the 2-factor guaranteed by the Petersen Theorem we have again an even regular graph. Thus, by induction, an even regular graph has a 2 -factorization. The following theorem allows to find several ( $\mathrm{s}, 1$ ) -VAT labelings of any even regular graph.
Let $G$ be a $2 r$-regular graph with vertices $x_{1}, x_{2} \ldots x_{n}$. Let s be an integer, $\mathrm{s} \in\left\{\left(\mathrm{r}_{\mathrm{n}}+1\right)(\mathrm{r}+1)+\mathrm{t}_{\mathrm{n}}: \mathrm{t}=0\right.$, $1, \ldots, r\}$. Then there exists an $(\mathrm{s}, 1)$-VAT labeling $\lambda$ of G such that $\lambda\left(\mathrm{x}_{\mathrm{i}}\right)=\mathrm{s}+(\mathrm{i}-1)$.

Proof : By induction on r . We show a stronger result. Not only we give an ( $s, 1$ ) -VAT labeling of G, but the vertex labels will be consecutive integers. Moreover, we can specify which weight $s+(i-1)$ will be assigned to which vertex by ordering the vertices xi accordingly. For $r=0$ the statement is trivial. The set of possible values of $s$ is $\{(0 n+1)(0+1)+0 n\}=\{1\}$. We label the vertices xi by $1+(i-1)=i$ for $i=1$, $2, \ldots, n$. In the inductive step we suppose the claim is true for p -regular graphs, $\mathrm{p}=0,1, \ldots, r$ We show that it is true also for $\mathrm{p}=\mathrm{r}+1$. Let G be a $(2 \mathrm{r}+2)$ regular graph with vertices $x_{1}, x_{2}, \ldots, x_{n}$. By the Petersen Theorem there exists a 2 -regular factor $F$ in $G$. By $\mathrm{G}^{\prime}$ we denote the the 2 r -regular graph obtained from $G$ by removing the edges of F . By the assumption $\mathrm{G}^{\prime}$ has an ( $\mathrm{s}, 1$ ) -VAT labeling $\lambda$ ' such that $\mathrm{s} \in\left\{\left(\mathrm{r}_{\mathrm{n}}+\right.\right.$ $\left.1)(r+1),\left(r_{n}+1\right)(r+1)+n, \ldots,\left(r_{n}+1\right)(r+1)+r_{n}\right\}$ and
the vertex labels are consecutive integers $\lambda^{\prime}\left(\mathrm{x}_{\mathrm{i}}\right)=\mathrm{k}+$ ( $\mathrm{i}-1$ ) for some k ( k is the smallest vertex label). The factor F is a collection of cycles. We order and orient these cycles arbitrarily. By out $\left(\mathrm{x}_{\mathrm{i}}\right)$ we denote the end vertex of the outgoing arc from $x_{i}$ and by in $\left(x_{i}\right)$ we denote the begin vertex of the incoming arc to $x i$.

We define the labeling $\lambda 1$ of $G$ by
$\lambda_{1}\left(\mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}}\right)=\lambda^{\prime}\left(\mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}}\right) \forall \mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}} \in \mathrm{E}\left(\mathrm{G}^{\prime}\right) \mathrm{i}, \mathrm{j} \in\{1,2, \ldots, \mathrm{n}\}$
$\lambda_{1}\left(\mathrm{x}_{\mathrm{i}}\right.$ out $\left.(\mathrm{xi})\right)=\lambda^{\prime}\left(\mathrm{x}_{\mathrm{i}}\right) \forall \mathrm{x}_{\mathrm{i}}$ out $\left(\mathrm{x}_{\mathrm{i}}\right) \in \mathrm{E}(\mathrm{F})$
$\lambda_{1}(\mathrm{xi})=\mathrm{n}(\mathrm{r}+2)+\mathrm{k}-\lambda^{\prime}\left(\right.$ in $\left.\left(\mathrm{x}_{\mathrm{i}}\right)\right) \forall \mathrm{xi} \in \mathrm{V}(\mathrm{F})$.
Another labeling $\lambda 2$ of $G$ is given by
$\lambda_{2}\left(\mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}}\right)=\lambda^{\prime}\left(\mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}}\right) \forall \mathrm{x}_{\mathrm{i}} \mathrm{x}_{\mathrm{j}} \in \mathrm{E}\left(\mathrm{G}^{\prime}\right) \mathrm{i}, \mathrm{j} \in\{1,2, \ldots, \mathrm{n}\}$
$\lambda_{2}\left(\mathrm{x}_{\mathrm{i}}\right.$ out $\left.\left(\mathrm{x}_{\mathrm{i}}\right)\right)=\mathrm{n}(\mathrm{r}+1)-(\mathrm{k}-1)+\lambda^{\prime}(\mathrm{xi}) \forall \mathrm{x}_{\mathrm{i}}$ out $\left(\mathrm{x}_{\mathrm{i}}\right)$ $\in \mathrm{E}$ (F)
$\lambda_{2}(\mathrm{xi})=2 \mathrm{k}+\mathrm{n}-1-\lambda^{\prime}\left(\mathrm{in}\left(\mathrm{x}_{\mathrm{i}}\right)\right) \forall \mathrm{x}_{\mathrm{i}} \in \mathrm{V}(\mathrm{F})$.

The weight of any vertex in labeling $\lambda_{1}$ is $w \lambda_{1}\left(x_{i}\right)=n(r$ $+2)+\mathrm{k}+\mathrm{s}+(\mathrm{i}-1)$. We know that $\mathrm{s}=\left(\mathrm{r}_{\mathrm{n}}+1\right)(\mathrm{r}+1)+$ $t_{n}$ for $t \in\{0,1, \ldots, r\}$. Since $k$ is the smallest vertex label and since we assign always an $n$-tuple of labels to either edges or vertices of a 2 -regular factor, we can write $\mathrm{k}=\mathrm{q}_{\mathrm{n}}+1=(\mathrm{q}-\mathrm{r}+\mathrm{r}) \mathrm{n}+1=(\mathrm{q}-\mathrm{r}) \mathrm{n}+\left(\mathrm{r}_{\mathrm{n}}+1\right)$. The value $t$ specifies that the $(\mathrm{r}+1-\mathrm{t})^{\mathrm{th}} \mathrm{n}$-tuple was used to label vertices in $\lambda^{\prime}$, thus $q \geq r-t$. We have $\mathrm{w} \lambda_{1}\left(\mathrm{x}_{\mathrm{i}}\right)=\mathrm{n}(\mathrm{r}+2)+(\mathrm{q}-\mathrm{r}) \mathrm{n}+\left(\mathrm{r}_{\mathrm{n}}+1\right)+\left(\mathrm{r}_{\mathrm{n}}+1\right)(\mathrm{r}+1)$ $+\mathrm{t}_{\mathrm{n}}+(\mathrm{i}-1)$
$=\mathrm{n}(\mathrm{r}+2)+\left(\mathrm{r}_{\mathrm{n}}+1\right)(\mathrm{r}+2)+(\mathrm{t}+\mathrm{q}-\mathrm{r}) \mathrm{n}+(\mathrm{i}-$ 1)

$$
=((r+1) n+1)(r+2)+(t+q-r) n+(i-1)
$$

The labeling $\lambda_{1}$ is an $\left(\mathrm{s}_{1}, 1\right)$-VAT labeling of the $2(\mathrm{r}+$ 1) -regular graph $G$ where the smallest weight is $s 1=$ $((r+1) n+1)(r+2)+(t+q-r) n$.

Not only both labelings $\lambda_{1}$ and $\lambda_{2}$ assign consecutive integers to the vertices, but also the weight of vertices always increases with the vertex subscript $i$ $\mathrm{w} \lambda_{1}\left(\mathrm{x}_{\mathrm{i}}\right)=\mathrm{s}_{1}+(\mathrm{i}-1) \wedge \mathrm{w} \lambda_{2}(\mathrm{xi})=\mathrm{s}_{2}+(\mathrm{i}-1)$.
This concludes the inductive step and the proof.

## Example:

We construct several (s, 1)-VAT labelings of $\mathrm{K}_{5}$ based on the construction given in the proof of Theorem 3.2. For $r=0$ is $G=K_{5}$ and the construction is trivial.


5 v5

$(1,1)-V A T$ labeling of $\bar{K}_{5}$.
Now for $r=1$ we take the factor $F$ to be the cycle $\mathrm{V}_{1} \mathrm{~V}_{2} \mathrm{~V}_{3} \mathrm{~V}_{4} \mathrm{~V}_{5} \mathrm{~V}_{1}$. We can construct a $(12,1)$-VAT labeling $\lambda_{1}$ of $\mathrm{C}_{5}$ or a $(17,1)$-VAT labeling $\lambda_{2}$ of $\mathrm{C}_{5}$. The weights are in bold.


A $(12,1)-$ VAT labeling and $a(17,1)-$ VAT labeling of $C_{5}$.

## III. CONCLUSION

We proposed magic labeling graph approach is used for solving VAT labeling graph. Due to additional restriction in TSP it is difficult to solve by using AP, even though online Assignment Problem solver fail to solve this problem. Based on the experiments, it can be concluded that the quality of solutions depends on the number of ants. The lower number of ants allows the individual to change the path much faster. The higher number of ants in population causes the higher accumulation of pheromone on edges, and thus an individual keeps the path with higher concentration of pheromone with a high probability. The final result differs from optimal solution by km (deviation is less than). The great advantage over the use of exact methods is that provides relatively good results by a comparatively low number of iterations, and is therefore able to find an acceptable solution in a comparatively short time, so it is useable for solving problems occurring in practical applications.

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