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Common Fixed Point Result for a Pair of Multi Valued Maps in a Partially Ordered Metric Space

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ABSTRACT

In this paper we prove a unique common fixed point theorem for a pair of multi-valued mappings satisfying a generalized (ξ,η) weak contractive condition in complete partially ordered metric space. Present paper is the improved version of result proved earlier in the literature.

Keywords: Fixed point , Partially ordered metric space , Multi valued mappings , Mappings ξ and η.

I. INTRODUCTION

The Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find those fixed points.

In 1969, Nadler [19] established the multi-valued version of Banach contraction principle. Later on , many fixed point theorems have been proved by various authors as generalization of the Nadler's theorem where the nature of contractive mapping is weakened along with some additional requirements, one may refer S. Reich(1972), , E. Zeidler (1985) I. Beg and A. Azam(1992), P.Z. Daffer (1995), P.Z. Daffer, H. Kaneko and W. Li(1996), C.Y. Qing(1996), Y. Feng and S. Liu(2006), D. Klim and D.Wardowski(2007).

Ran and Reurings established the existence of unique fixed point for single valued mapping in partially ordered metric spaces. Their result was further extended by A. Petrusel and I.A. Rus (2005), J.J. Nieto and R. Rodriguez-Lopez (2005), T.G. Bhaskar and V. Lakshmikantham (2006), J.J. Nieto and R. Rodriguez-Lopez(2007), D. O'Regan and A. Petrusel (2008) I. Beg and A.R. Butt (2009), J. Harjani, K. Sadarangani (2009), I. Beg and A.R. Butt(2010). Further Bhaskar and Lakshmikantham studied the existence and uniqueness of a coupled fixed point in partially ordered metric space with assumption that the single valued mapping satisfies the weaker contraction condition.

The aim of this paper is to improve, extend and unify the above existing results and prove common fixed point theorem for a pair of multi valued mappings satisfying a generalized (ξ,η) weak contractive condition in complete partially ordered metric space.

Definition 1.1: Let (X,d) be a metric space and let $B(X)$ be the class of all nonempty bounded subsets of X. We define the functions $\delta: B(X)\times B(X)\to R^+$ and $D: B(X)\times B(X)\to R^+$ as follows:

$$\delta(A,B)=\sup\{d(a,b): a\in A, b\in B\}$$
$$D(A,B)=\inf\{d(a,b): a\in A, b\in B\}$$

where R^{+} denotes the set of all positive real numbers. For $\delta(\{a\},B)$ and $\delta(\{a\},\{b\})$, we write $\delta(a,B)$ and $d(a,b)$ respectively. Clearly , $\delta(A,B)=\delta(B,A)$. We appeal to the fact that $\delta(A,B)=0$ if and only if $A=B=\{x\}$, for $A,B\in B(X)$ and $0\le\delta(A,B)\le\delta(A,C)+\delta(C,B)$, for all $A,B,C\in B(X)$.

A point $x\in X$ is called a fixed point of T if $x\in Tx$. If there exists a point $x\in X$ such that $Tx=\{x\}$, then x is called the end point of the mapping *T.*

Definition1.2: A partial order relation is a binary relation $\le$ on X which satisfies the following conditions:

i) $x\le x$ (reflexivity),

ii) if $x\le y$ and $y\le x$ then $x=y$ (anti symmetry),

iii) if $x\le y$ and $y\le z$ then $x\le z$ (transitivity),

for all $x,y,z\in X$

A set with a partial order relation $\le$ is called a partially ordered set.

Definition1.3: Let $(X,\le)$ be a partially ordered set and $x,y\in X$. Elements x and y are said to be comparable elements of X if either $x\le y$ or $y\le x$.

Definition1.4: Let $(X,\le)$ be a partially ordered set and $'d'$ be a metric defined on X then (X,d) is called a partially ordered metric space.

Definition1.5: A function $\xi:[0,\infty)\to[0,\infty)$ is called an altering distance function if it is Continuous , Monotonically non-decreasing and $\xi(t)=0$ if and only if $t=0$.

Theorem 2.1: Let $(X,\le)$ be a partially ordered set and suppose that there exists a metric $'d'$ in X such that (X,d) is a complete metric space. Let $F,G:X\to B(X)$ be two multi valued mappings such that the following conditions are satisfied :

i) There exists $x_0\in X$ such that $x_1=Fx_0\le GFx_0=Gx_1=x_2=Gx_1\le FGx_1=Fx_2=x_3$

ii) If $x_n\to x$ is a non-decreasing sequence in X , then $x_n\le x$, for all n.

iii) $\xi(\delta(Fx,Gy))\le\xi\Big\{\max.\Big[d(x,y),D(x,Fx),D(y,Gy),\frac{D(x,Gy)+}{2}$

$$-\eta\{\max.[d(x,y),\delta(x,Fx),\delta(y,Gy)]\}$$

for all comparable $x,y\in X$, where ξ is an altering distance function and $\eta:[0,\infty)\to[0,\infty)$ is any continuous function with $\eta(t)=0$ if and only if $t=0$.

Then F and G will have a common fixed point in *X.*

Proof: Let x_0 be an arbitrary point of X. We can define a sequence $\{x_n\}$ in X as follows:

$x_{2n+1}=Fx_{2n}$ and $x_{2n+2}=Gx_{2n+1}$, for $n\in\{0,1,2,----\}$, where the successive terms of $\{x_n\}$ are all different.

Since we have , $x_1=Fx_0\le GFx_0=Gx_1=x_2=Gx_1\le FGx_1=Fx_2=x_3$

Continuing this process , we have , $x_1\le x_2\le x_3\le ---\le x_n\le x_{n+1}\le ---$

Using the monotone property of ξ and condition (iii) , we have for all $n\ge 0$

$$\xi\left(d\left(x_{n+1},x_{n+2}\right)\right)\le\xi\left(\delta\left(Fx_n,Gx_{n+1}\right)\right)$$

$$\le\xi\left\{\max.\left[d(x_n,x_{n+1}),D(x_n,Fx_n),D(x_{n+1},Gx_{n+1}),\frac{D(x_n,Gx_{n+1})+D(x_{n+1},Fx_n)}{2}\right.\right.$$

$$-\eta\{\max.[d(x_n,x_{n+1}),\delta(x_n,Fx_n),\delta(x_{n+1},Gx_{n+1})]\}$$

$$\le\xi\left\{\max.\left[d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),\frac{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}{2}\right]\right\}$$

$$-\eta\{\max.[d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})]\}$$

--------- (2.1.1)

Since $\frac{d(x_n, x_{n+2})}{2} \le \max.[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]$

Therefore from (2.1.1) , we can write ,

$\xi(d(x_{n+1}, x_{n+2})) \le \xi\{\max.[d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})]\} - \eta\{\max.[d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})]\}$ ----- (2.1.2)

If $d(x_n, x_{n+1}) \le d(x_{n+1}, x_{n+2})$, for some positive integer 'n' .

Then from (2.1.2) , we have ,

$\xi(d(x_{n+1}, x_{n+2})) \le \xi(d(x_{n+1}, x_{n+2})) - \eta(d(x_{n+1}, x_{n+2}))$

i.e. $\eta(d(x_{n+1}, x_{n+2})) \le 0$, which implies that $d(x_{n+1}, x_{n+2}) = 0$ or $x_{n+1} = x_{n+2}$, which contradicts to our assumption that $x_n \ne x_{n+1}$, for each n.

$\therefore$ $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for all $n \ge 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non- negative real numbers. Hence there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ --- (2.1.3)

$\therefore$ From (2.1.2) we will have for all $n \ge 0$,

$\xi(d(x_{n+1}, x_{n+2})) \le \xi(d(x_n, x_{n+1})) - \eta(d(x_n, x_{n+1}))$

Taking limit as $n \to \infty$, using (2.1.3) and the continuities of functions ξ and η , we get

$\xi(r) \le \xi(r) - \eta(r)$, which is a contradiction unless $r = 0$.

Hence,

$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ --(2.1.4)

Now , we show that $\{x_n\}$ is a Cauchy sequence .

We prove this by method of contradiction i.e. we assume that $\{x_n\}$ is not a Cauchy sequence.

$\therefore$ There exists an $\in > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \ge \in$

Assuming that $n(k)$ is the smallest such positive integer , we get $n(k) > m(k) > k$, $d(x_{m(k)}, x_{n(k)}) \ge \in$ and $d(x_{m(k)}, x_{n(k)}) < \in$

Now , $\in \le d(x_{m(k)}, x_{n(k)}) \le d(x_{m(k)}, x_{n(k-1)}) + d(x_{n(k-1)}, x_{n(k)})$

i.e. $\in \le d(x_{m(k)}, x_{n(k)}) < \in + d(x_{n(k-1)}, x_{n(k)})$

Taking limit as $k \to \infty$ in the above inequality and using (2.1.4) ,

we have , $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \in$ --------------(2.1.5)

Again , $d(x_{m(k)}, x_{n(k)}) \le d(x_{m(k)}, x_{m(k+1)}) + d(x_{m(k+1)}, x_{n(k+1)}) + d(x_{n(k+1)}, x_{n(k)})$

and $d(x_{m(k+1)}, x_{n(k+1)}) \le d(x_{m(k+1)}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k+1)})$

Taking limit as $k \to \infty$ in the above inequalities , using (2.1.4) and (2.1.5) , we have ,

$\lim_{k\to\infty} d(x_{m(k+1)}, x_{n(k+1)}) = \in$ --------- (2.1.6)

Also , $d(x_{m(k)}, x_{n(k)}) \le d(x_{m(k)}, x_{n(k+1)}) + d(x_{n(k+1)}, x_{n(k)})$

and $d(x_{m(k)}, x_{n(k+1)}) \le d(x_{m(k}, x_{n(k)}) + d(x_{n(k)}, x_{n(k+1)})$

Taking limit as $k \to \infty$ in the above inequalities and using (2.1.4) , (2.1.5) , we get ,

$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k+1)}) = \in$ -------- (2.1.7)

Similarly , we will have ,

$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k+1)}) = \in$ -------(2.1.8)

For each positive integer 'k' , $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ξ and the condition (iii) , we have ,

$$\xi(d(x_{m(k+1)}, x_{n(k+1)})) \le \xi(\delta(Fx_{m(k)}, Gx_{n(k)}))$$

$$\le \xi\left\{\max.\left[d(x_{m(k)}, x_{n(k)}), D(x_{m(k)}, Fx_{m(k)}), D(x_{n(k)}, Gx_{n(k)}), \frac{D(x_{m(k)}, Gx_{n(k)}) + D(x_{n(k)}, Fx_{m(k)})}{2}\right]\right\}$$

$$-\eta\{\max.[d(x_{m(k)}, x_{n(k)}), \delta(x_{m(k)}, Fx_{m(k)}), \delta(x_{n(k)}, Gx_{n(k)})]\}$$

$$\leq \xi\left\{\max.\left[d\left(x_{m(k)},x_{n(k)}\right),d\left(x_{m(k)},x_{m(k+1)}\right),d\left(x_{n(k)},x_{n(k+1)}\right),\frac{d\left(x_{m(k)},x_{n(k+1)}\right)+d\left(x_{n(k)},x_{m(k+1)}\right)}{2}\right]\right\}$$
$$-\eta\{\max.[d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{m(k+1)}),d(x_{n(k)},x_{n(k+1)})]\}$$

Taking $k\to\infty$, using (2.1.5) , (2.1.6) , (2.1.7) , (2.1.8) and using the continuities of ξ and η,

We have , $\xi(\in)\leq\xi(\in)-\eta(\in)$, which is a contradiction by virtue of a property of η.

Hence $\{x_n\}$ is a Cauchy sequence , and since X is a complete metric space , there exists a point $u\in X$ such that $x_n\to u$ as $n\to\infty$.

i.e. $\lim_{n\to\infty} d(x_n,u)=d(u,u)=0$, $\lim_{n,m\to\infty} d(x_n,x_m)=0$

By the assumption (ii) $x_n\leq u$, for all *n*.

$\therefore$ By monotone property of ξ and by substituting $x=x_{2n+1}$, $y=Gu$ in (iii) , we have ,

$\xi(d(x_{2n+1},Gu))\leq\xi(\delta(Fx_{2n},Gu))$

$$\leq\xi\left\{\max.\left[d(x_{2n},u),D(x_{2n},Fx_{2n}),D(u,Gu),\frac{D(x_{2n},Gu)+D(u,Fx_{2n})}{2}\right]\right\}$$
$$-\eta\{\max.[d(x_{2n},u),\delta(x_{2n},Fx_{2n}),\delta(u,Gu)]\}$$
$$\leq\xi\left\{\max.\left[d(x_{2n},u),D(x_{2n},x_{2n+1}),D(u,Gu),\frac{D(x_{2n},Gu)+D(u,x_{2n+1})}{2}\right]\right\}$$
$$-\eta\{\max.[d(x_{2n},u),\delta(x_{2n},x_{2n+1}),\delta(u,Gu)]\}$$

Taking limit as $n\to\infty$, we get ,

$\xi(\delta(u,Gu))\leq\xi\left[D(u,Gu),\frac{D(u,Gu)}{2}\right]-\eta[\delta(u,Gu)]$

i.e. $\xi(\delta(u,Gu))\leq\xi(\delta(u,Gu))-\eta(\delta(u,Gu))$, which is a contradiction unless $\delta(u,Gu)=0$,

i.e. $\{u\}=Gu$, i.e. $'u'$ is a fixed point of *G*.

Now , we show that $'u'$ is also a fixed point of *F*.

$\therefore$ Substitute $x=x_{2n+2}$, $y=Fu$ in (iii) , we have ,

$\xi(d(x_{2n+2},Fu))\leq\xi(\delta(x_{2n+2},Fu))\leq\xi(\delta(Gx_{2n+1},Fu))$

$$\leq\xi\left\{\max.\left[d(u,x_{2n+1}),D(u,Fu),D(x_{2n+1},Gx_{2n+1}),\frac{D(u,Gx_{2n+1})+D(x_{2n+1},Fu)}{2}\right]\right\}$$
$$-\eta\{\max.[d(u,x_{2n+1}),\delta(u,Fu),\delta(x_{2n+1},Gx_{2n+1})]\}$$

On taking limit as $n\to\infty$, we get

$\xi(\delta(u,Fu))\leq\xi\left[D(u,Fu),\frac{D(u,Fu)}{2}\right]-\eta[\delta(u,Fu)]$

i.e. $\xi(\delta(u,Fu))\leq\xi(\delta(u,Fu))-\eta(\delta(u,Fu))$, which is again a contradiction unless $\delta(u,Fu)=0$,

which implies that $\{u\}=Fu$, i.e. $'u'$ is a fixed point of *F*.

Hence $'u'$ is a common fixed point of *F* and *G*.

Corollary 2.2: Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric $'d'$ in X such that (X,d) is a complete metric space. Let $F:X\to B(X)$ be a multi valued mapping such that the following conditions are satisfied :

i) There exists $x_0\in X$ such that $x_1=Fx_0$

ii) If $x_n\to x$ is a non-decreasing sequence in X , then $x_n\leq x$, for all n.

iii) $$\xi(\delta(Fx,Fy))\leq\xi\left\{\max.\left[d(x,y),D(x,Fx),D(y,Fy),\frac{D(x,Fy)+D(y,Fx)}{2}\right]\right\}$$
$$-\eta\{\max.[d(x,y),\delta(x,Fx),\delta(y,Fy)]\}$$

for all comparable $x, y \in X$, where ξ is an altering distance function and $\eta : [0,\infty) \to [0,\infty)$ is any continuous function with $\eta(t) = 0$ if and only if $t = 0$.

Then F has a fixed point in *X*.

Proof: If we substitute $F = G$ in Theorem 2.1 then we get the proof of Corollary 2.2 which is same as the result proved in [3].

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