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# Equilibrium equations for thermal buckling analysis of annular plates

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## ABSTRACT

In the following paper equations for buckling of annular plates which are made of functionally graded material are derived when subjected to temperature load. Equilibrium equations are derived using first order shear deformation theory under the thermal loads. The fundamental partial differential equations are derived using minimum potential energy. The material properties are assumed to be varying as a power form of the thickness coordinate variable z.

These equations are solved by using number of methods like energy methods, analytical methods, finite difference method, and finite element methods.

**Keywords:** Functionally graded materials, FSDT, Buckling

## I. INTRODUCTION

Composite materials are cast using two or more materials having different physical or chemical properties. Fiber reinforced composite materials come under the category of high performance products. They are light but strong enough to take harsh loadings. Their use over the years has expanded into many areas like aerospace components, automotive and marine industries etc. Only shortcoming with these materials is the interface of the two materials across which there is a mismatch in mechanical properties causing large inter-laminar stresses. When these kinds of materials are exposed to high temperature environment then there arises the problem of debonding and delamination problems. Cracks develop slowly at the interfaces and grow into weaker material sections.

To overcome the problem of debonding and delamination, group of scientists from Japan in 1984 introduced a new material called Functionally graded materials.

Functionally graded materials are the materials which are not homogeneous and material properties vary smoothly from one surface to the other. The constituent materials volume fraction is gradually varied to obtain varying properties. This variation in composition yields us the FGM's with graded properties. The gradation in properties of the material causes temperature stresses, residual stresses, and stress concentration factors to reduce. For a high temperature environment these materials are made of ceramic and metals or from a combination of different materials. The ceramic constituent of the material has high temperature resistance. On the other hand ductile metal constituent is fracture resistant. These fractures are because of stresses due to high temperature. Ceramic and metal combination can be easily manufactured. Using graded property materials the interface problems of composite materials are removed and stress distributions are smooth.

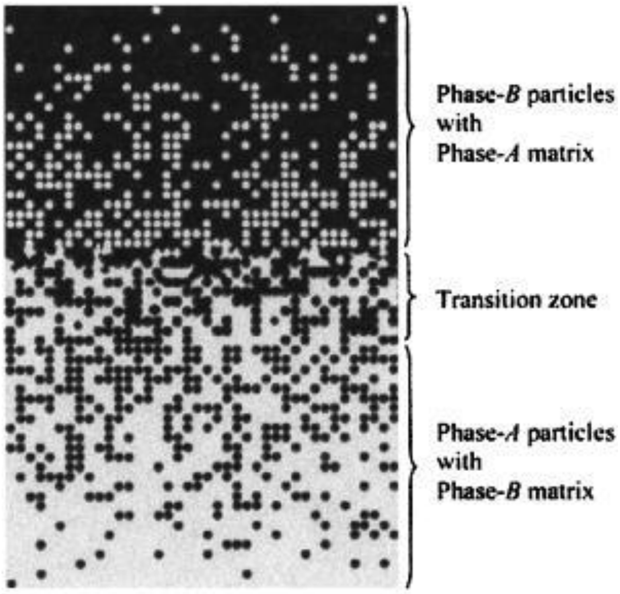
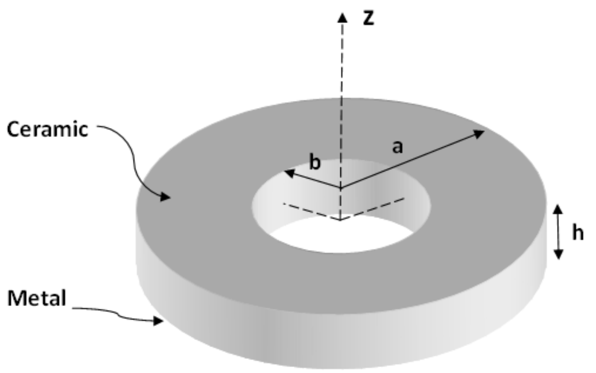



Figure 1 Functionally graded material

## II. EQUILIBRIUM EQUATIONS




An annular plate with outer radius a, inner radius b, and thickness h made of functionally graded material is considered.

The material properties of the plate vary along the thickness of the plate. The coordinate axis across the plate thickness is taken as z. So the functional relationships of E and $\alpha$ with respect to z for the plate are

$$E = E(z) = E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^P$$

$$\alpha = \alpha(z) = \alpha_m + (\alpha_c - \alpha_m)\left(\frac{2z+h}{2h}\right)^P$$

Where,

Em=Modulus of elasticity of metal,

Ec= Modulus of elasticity of ceramic,

$\vartheta$ = Poisson's ratio of FGM plate (assumed constant)

P=Volume fraction exponent which takes values greater than or equal to zero

$\alpha_m$-Coefficient of thermal expansion of metal

$\alpha_c$-coefficient of thermal expansion of ceramic

The power law assumption will ensure simple rule of mixtures. This rule of mixtures applies only to the thickness direction. First order shear deformation theory is employed in the following study because it includes the effects of shear deformation.

A) Displacement field

Assuming a displacement field that allows shear deformation,

$$u_0(r,z) = u(r) + z\beta_r(r)$$

$$w_0(r,z) = w(r)$$

u0, and w0 - displacements of a point $(r, \theta, z)$ in the x and z directions respectively.

u - in-plane displacements of a point $(r, \theta)$ on the middle plane

w - transverse displacements of a point $(r, \theta)$ on the middle plane

$\beta_r$ - rotations of the normal to the middle plane about '$\theta$' axes

Strain-displacement relationship

The strains at any point (r, z), in terms of strain and curvature of middle plane are[14]

$$\epsilon_r = \bar{\epsilon}_r + zk_r$$

$$\epsilon_\theta = \bar{\epsilon}_\theta + zk_\theta$$

The relationship between the middle plane strains and the middle surface displacements are,

$$\bar{\epsilon}_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \quad \bar{\epsilon}_\theta = \frac{u}{r}$$

$$k_r = \frac{\partial \beta_r}{\partial r}, \qquad k_\theta = \frac{\beta_r}{r}$$

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + z\frac{\partial \beta_r}{\partial r}$$

$$\epsilon_\theta = \frac{u}{r} + z\frac{\beta_r}{r}$$

$$\gamma_{rz} = \beta_r + \frac{\partial w}{\partial r}$$

#### B) Stress-Strain relationships

Stresses developed are given by the following equations

$$\sigma_r = \frac{E(z)}{(1-\vartheta^2)}[\epsilon_r + \vartheta\epsilon_\theta - (1+\vartheta)\alpha T]$$

$$\sigma_\theta = \frac{E(z)}{(1-\vartheta^2)}[\epsilon_\theta + \vartheta\epsilon_r - (1+\vartheta)\alpha T]$$

$$\tau_{rz} = \frac{E(z)}{2(1+\vartheta)}\gamma_{rz}$$

#### C) Stress resultants and stress couples

The forces and moments $N_i, M_i$ and $Q_r$ of axisymmetric circular plates arising out of stresses are written as,

$$(N_i, M_i) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i(1,z)dz\,, \quad \text{i=r},\theta.$$

$$Q_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{rz}dz\,,$$

Therefore,

$$N_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_r\,dz \qquad N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta\,dz$$

$$M_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_r)z\,dz \qquad M_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_\theta)z\,dz$$

Where,

$N_r$ and $N_\theta$ - radial and circumferential in-plane force resultants and

$M_r$ and $M_\theta$ - radial and circumferential moments-resultants (stress couples).

#### D) Equilibrium equations and Natural boundary conditions.

To derive equations of equilibrium minimum potential energy is used. These equations and boundary conditions are presented in the following section.

The potential energy Π for the plate element is defined as

$$\Pi = U + V - W_{er}$$

Where,

U= Strain energy of the plate

V= Potential energy due to loads

Wer= Work done by edge stress on edge 'r'

The principle of virtual displacements can be expressed as

$$\delta\Pi = 0$$

The total strain energy is

$$\pi = U + V - W_{er}$$

$$\delta\pi = \delta U + \delta V - \delta W_{er}$$

$$= \int_\theta \left[(rN_r - N_{r0_r})\delta u + (rM_r - \bar{M}_r)\delta\beta_r + \left(Q_r r - \bar{Q}_r + rN_{r0}\frac{\partial w}{\partial r}\right)\delta w\right]d\theta$$
$$+ \int_r \int_\theta \left\{\left[N_\theta - \left(N_r + r\frac{\partial N_r}{\partial r}\right)\right]\delta u + \left[M_r - \left(M_r + r\frac{\partial M_r}{\partial r}\right) + rQ_r\right]\delta\beta_r + \left[-\frac{\partial\left(rN_{r0}\frac{\partial w}{\partial r}\right)}{\partial r} - \left(Q_r + r\frac{\partial Q_r}{\partial r}\right) - rq\right]\delta w\right\} dr d\theta$$

The equations of equilibrium and consistent boundary conditions are obtained by setting the individual integral terms in the above equation to zero

$$\delta u: \quad \frac{\partial N_r}{\partial r} + \frac{(N_r - N_\theta)}{r} = 0$$

$$\delta \beta_r: \quad \frac{\partial M_r}{\partial r} + \frac{(M_r - M_\theta)}{r} - Q_r = 0$$

$$\delta w: \quad Q_r + r\frac{\partial Q_r}{\partial r} = -N_{r0}\left(r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right) - rq$$

And natural boundary conditions are,

On the edge 'r'

$$u: \quad N_{r0} = rN_r$$

$$\beta_r: \quad \bar{M}_r = rM_r$$

$$w: \quad \bar{Q}_r = rQ_r - rN_{r0}\frac{\partial w}{\partial r}$$

E. Equilibrium equations in terms of displacement functions

$$A_{11}\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} - \frac{u}{r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + r\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\right) + B_{11}\left(r\frac{\partial^2 \beta_r}{\partial r^2} + \frac{\partial \beta_r}{\partial r} - \frac{\beta_r}{r}\right) - A_{12}\left(\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right) = 0$$

$$B_{11}\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} - \frac{u}{r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + r\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\right) + D_{11}\left(r\frac{\partial^2 \beta_r}{\partial r^2} + \frac{\partial \beta_r}{\partial r} - \frac{\beta_r}{r}\right) - B_{12}\left(\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right) - A_{66}\left(r\beta_r + r\frac{\partial w}{\partial r}\right) = 0$$

$$A_{66}\left(r\frac{\partial \beta_r}{\partial r} + \beta_r + \frac{\partial w}{\partial r} + r\frac{\partial^2 w}{\partial r^2}\right) = -N_{r0}\left(r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right) - rq$$

## III. RESULTS

Thus the equilibrium equations for buckling of annular plates for temperature are derived using minimum potential energy.

The above equations can be solved by number of methods like energy methods, analytical methods, finite difference method, and finite element methods.

## IV. REFERENCES


[1] Aghelinejad, M., Zare, K., Ebrahimi, F. and Rastgoo, A. (2011) "Nonlinear thermo-mechanical post-buckling analysis of thin functionally graded annular plates based on Von-Karman's plate theory" Mechanics of Advanced Materials and Structures, Volume 18, Issue 5, pages 319-326.

[2] Dong, C.Y. (2008) "Three-dimensional free vibration analysis of functionally graded annular plates using the Chebyshev–Ritz method" Materials and Design, Volume 29, pages 1518–1525.

[3] Golmakani, M.E., and Kadkhodayan, M. (2011) "Nonlinear bending analysis of annular FGM plates using higher-order shear deformation plate theories" Composite Structures, Volume 93, pages 973–982.

[4] Hamad, M.H., and Tarlochan, F. (2013) "An axisymmetric bending analysis of functionally graded annular plate under transverse load via generalized differential quadrature method" International Journal of Research in Engineering and Technology, Volume 02, Issue 07, Jul-2013.

[5] Hosseini-Hashemi, Sh., Fadaee, M. and Es'haghi, M. (2010) "A novel approach for in-plane/out-of-plane frequency analysis of functionally graded circular/annular plates" International Journal of Mechanical Sciences, Volume 52, pages 1025–1035.

[6] Iman Davoodi Kermani., Mostafa Ghayour and Hamid Reza Mirdamadi. (2012) "Free vibration analysis of multi-directional functionally graded circular and annular plates" Journal of Mechanical Science and Technology, Volume 26 (11), pages 3399-3410.

[7] Koohkan, H., Kimiaeifar, A., Mansourabadi, A. and Vaghefi, R. (2010) "An analytical approach on the buckling analysis of circular, solid and annular functionally graded thin plates" Journal of Mechanical Engineering, Volume. ME 41, Issue No. 1.

[8] Lei, Zheng. and Zheng, Zhong. (2009) "Exact Solution for Axisymmetric Bending of Functionally Graded Circular Plate" Tsinghua science and technology, Volume 14, Number S2, pages 64-69.

[9] Li, Shi-Rong., Zhang, Jing-Hua. and Zhao, Yong-Gang. (2007) "Nonlinear thermomechanical post-buckling of circular FGM plate with geometric imperfection" Thin-Walled Structures, Volume 45, pages 528–536.

[10] Ma, L.S. and Wang, T.J. (2003) "Nonlinear bending and post-buckling of a functionally graded circular

plate under mechanical and thermal loadings.” International Journal of Solids and Structures, Volume 40, pages 3311–3330.

[11] Ma, L.S. and Wang, T.J. (2004) “Relationships between axisymmetric bending and buckling solutions of FGM circular plates based on third-order plate theory and classical plate theory” International Journal of Solids and Structures, Volume 41, pages 85–101.

[12] Najafizadeh, M. M. and Eslami, M. R. (2002a) “Thermoelastic stability of orthotropic circular plates” Journal of Thermal Stresses, Volume 25, Issue10, pages 985-1005.

[13] Najafizadeh, M.H. and Eslami, M.R. (2002b) "Buckling analysis of circular plates of functionally graded materials under uniform radial compression." International Journal of Mechanical Sciences, Volume 44, Issue 12, December 2002, Pages 2479-2493.

[14] Najafizadeh, M.M. and Hedayati, B. (2004) “Refined theory for thermoelastic stability of functionally graded circular plates” Journal of Thermal Stresses, Volume 27, Issue 9, pages 857-880.

[15] Najafizadeh, M.M. and Heydari, H.R. (2008) “An exact solution for buckling of functionally graded circular plates based on higher order shear deformation plate theory under uniform radial compression” International Journal of Mechanical Sciences, Volume 50, pages 603–612.

[16] Nie, G.J. and Zhong, Z. (2007a) “Semi-analytical solution for three-dimensional vibration of functionally graded circular plates” Comput. Methods Appl. Mech. Engrg., Volume 196, pages 4901–4910.

[17] Nie, G.J. and Zhong, Z. (2007b) “Axisymmetric bending of two-directional functionally graded circular and annular plates” Acta Mechanica Solida Sinica, Volume 20, issue 4, December, 2007.

[18] Nie, G.J. and Zhong, Z. (2010) “Dynamic analysis of multi-directional functionally graded annular plates” Applied Mathematical Modelling, Volume 34, pages 608–616.

[19] Prakash, T. and Ganapathi, M. (2006) “Asymmetric flexural vibration and thermoelastic stability of FGM circular plates using finite element method” Composites: Part B, Volume 37, pages 642–649.

[20] Reddy, J.N., Wang, C.M. and Kitipornchai, S. (1999) "Axisymmetric bending of functionally graded circular and annular plates” European Journal of Mechanics - A/Solids, Volume 18, Issue 2, March-April 1999, Pages 185-199.

[21] Sahraee, S. and Saidi, A.R. (2009) “Axisymmetric bending analysis of thick functionally graded circular plates using fourth-order shear deformation theory” European Journal of Mechanics A/Solids, Volume 28, pages 974–984.

[22] Saidi, A.R. and Hasani Baferani, A. (2010) “Thermal buckling analysis of moderately thick functionally graded annular sector plates” Composite Structures, Volume 92, pages 1744–1752.

[23] Saidi, A.R., Rasouli, A. and Sahraee, S. (2009) “Axisymmetric bending and buckling analysis of thick functionally graded circular plates using unconstrained third-order shear deformation plate theory” Composite Structures, Volume 89, pages 110–119.

[24] Serge Abrate (2006) “Free vibration, buckling, and static deflections of functionally graded plates” Composites Science and Technology, Volume 66 pages 2383–2394.

[25] Shen, Hui-Shen (2007) “Thermal post buckling behavior of shear deformable FGM plates with temperature-dependent properties” International Journal of Mechanical Sciences, Volume 49, pages 466–478.

[26] Shi, P. and Dong, C.Y. (2012) “Vibration analysis of functionally graded annular plates with mixed boundary conditions in thermal environment” Journal of Sound and Vibration, volume 331, pages 3649–3662.