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Stability of L −periodic equilibrium solutions of Navier-Stokes equations on infinite strip

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ABSTRACT

In this paper, we assume that a smooth equilibrium solution U_0, p_0 of Navier-Stokes equations is given on an infinite strip $\Omega = IR \times]-\frac{1}{2},\frac{1}{2}[$, the problem of stability that arises in the infinite plate ($\Omega = IR^2 \times]-\frac{1}{2},\frac{1}{2}[$) disappears in our case using the same tools in [7].

Keywords : Navier-Stokes equations, Fourier series, Stability, Regularity

I. Notations and Preliminaries

For $\mathcal{X},\mathcal{Y}$ Banach spaces, $\|.\|_{\mathcal{X}}, \|.\|_{\mathcal{Y}}$ are their respective norms. $L(\mathcal{X},\mathcal{Y})$ is the space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ with $\| T \|$ the operator norm.

For A a linear operator on $\mathcal{X}$ and $E \subseteq \mathcal{X}$ a subspace, $A|_E$ is the restriction of A to E.

For any $\Omega, H^p(\Omega)$ is the Sobolev space of functions having square integrable derivatives up to order p with $(.,.)_p$ and $\|.\|_{H^p(\Omega)}$ the usual scalar product and norm on $H^p(\Omega)$ .We set $\mathcal{L}^2(\Omega) = H^0(\Omega)$ and $\|.\|_{H^p} = \|.\|_{H^p(\Omega)}$ and extend this notation to vectors and set :

$$\| u \|_{\mathcal{L}^2}^2 = \| u_1 \|_{\mathcal{L}^2}^2 + \| u_2 \|_{\mathcal{L}^2}^2$$

where $u = (u_1, u_2) \in (\mathcal{L}^2(\Omega))^2$, Likewise with the Sobolev norms. The scalar product on $(H^p(\Omega))^2$ is $\langle .,. \rangle_p$, with :

$$\langle u, v \rangle_p = \sum_{i=1}^{2} (u_i, v_i)_p, u_i, v_i \in H^p(\Omega)$$

where $u = (u_1, u_2), v = (v_1, v_2)$
and we set $\langle .,. \rangle = \langle .,. \rangle_0$.

$C^p(\bar{\Omega})$ is the space of functions p times continously differentiable on $\bar{\Omega}$ and $C_0^p(\bar{\Omega})$ is the space of functions $f \in C^p(\bar{\Omega})$ with supp$f$ compact.

II. Introduction

Consider the Navier-Stokes equation on an infinite strip $\Omega = \mathrm{IR} \times]-\frac{1}{2},\frac{1}{2}[$:

$$\frac{\partial U}{\partial t} = \nu \Delta U - (U.\nabla)U + \nabla p + f, \operatorname{div} U = 0 \quad (1)$$

Where $U = (u_1, u_2)$ satisfies Dirichlet boundary conditions at $y = \pm 1/2$, p is the pressure, ν kinematic viscosity and f a time independent outer force.

We assume that a smooth equilibrium solution $U_0 = (u_1, u_2), p_0$ of equation (1) is given, which is $L-$periodic in x (i.e. $u_0(x + \alpha L, y) = u_0(x, y)$ Such that $\alpha \in \mathbb{Z}$) and likewise with p_0.

We can study The stability of $U_0 = (u_1, u_2), p_0$ against small perturbations under two aspects:
(i) The perturbations are L -periodic in x.
(ii)The perturbations are in $(\mathcal{L}^2(\Omega))^2$.

The relation between (i) and (ii) is the mathematical tools used by physicists in connection with Schrödinger equations with periodic potentials [4]. We will use the same tools used before like the direct integrals (see [1] , [4] , [7]) and the Θ-Periodic functions (ie. generalisation of periodic function).

To discuss the stability of U_0, p_0 with respect to various classes of perturbations we replace U, p in (1) by $U_0 + v, p_0 + \pi$ and we use that U_0, p_0 is an equilibrium solution of (1), Whereby it is supposed that v satisfy the same Dirichlet boundary conditions. By a straightforward computation we obtain:

$$\begin{cases} \frac{\partial v}{\partial t} = \nu\Delta v - (u_0.\nabla)v - (v.\nabla)u_0 - (v.\nabla)v - \nabla\pi \\ \text{div } v = 0, v/\,\partial\Omega = 0 \end{cases} \quad (2)$$

Now let's put this problem in a functional analytic frame.

Set $V = \{f \in (H_0^1(\Omega))^2, \text{div } f = 0\}$ and $E = \overline{V}^{(\mathcal{L}^2(\Omega))^2}$, and $\mathcal{P}$ the orthogonal projection onto E

We apply $\mathcal{P}$ to the both sides of (2) we get:

$$\partial_t v = (A_s + \mathcal{P}T_0)v + \mathcal{P}N(v) \quad (3)$$

Where A_s is the Stokes operator defined by:

$$f \in D(A_S) \text{ iff } \begin{cases} f \in (H^2(\Omega) \cap H_0^1(\Omega))^2, \text{div } f = 0 \\ A_S f = \nu\mathcal{P}\Delta f, \forall f \in D(A_S) \end{cases}$$

T_0 the linear operator defined by

$$T_0 f = -(u_0.\nabla)f - (f.\nabla)u_0, \forall f \in (H^1(\Omega) \cap H_0^1(\Omega))^2.$$

N is no-linear operator defined on E by $N(v) = -(v.\nabla)v$ and let $B = A_s + \mathcal{P}T_0$.

Proposition 2.1 [7]

$$E = \{v \in L^2(\Omega), \text{div } v = 0 \backslash v \perp \nabla p, \forall p \in H^1(\Omega)\}$$

Remark 1

$$E^{\perp} \supset \{\nabla\varphi, \varphi \in H^1(\Omega)\}$$

We have $\pi \in H^1(\Omega)$ then according to the previous remark $\nabla\pi \in E^{\perp}$, and on the other hand we have for all $v \in E, \partial_t v \in E$. cause ∂_t commute with v.

Proposition 2.2

A_S is selfadjoint and $\leq -\varepsilon$ for some $\varepsilon > 0$, and exists $c > 0$ such if $A_S v = f$ for $v \in D(A_S)$ and $f \in E$, then $\|v\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$ (*)

Proof

We have $\langle A_S v, v\rangle \leq \frac{-\nu}{c(\Omega)}\|v\|_{L^2}^2$, where $c(\Omega)$ is the Poincaré constant, then it suffice to take $\varepsilon = \frac{\nu}{c(\Omega)}$.
$\langle -A_S v, v\rangle \geq \varepsilon\|v\|_{L^2}$ so $\|A_S v\|_{L^2} \geq \|v\|_{L^2}$ which implies A_S is injective .

Finaly, A_S is bijective from $D(A_S)$ to $Im(A_S)$.

Corollary 1

T_0 is relatively bounded with respect to A_S ie : for all $\delta > 0$ there is $K_\delta > 0$ such that:

$$\|T_0 u\|_{L^2} \leq \delta\|A_S u\|_{L^2} + K_\delta\|u\|_{L^2}, \forall u \in D(A_S)$$

Proof

Set A operator on $L^2(\Omega)$ such that $(Au)_i = \Delta u_i$ for all u in D(A) with $D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^2, i = 1,2$.
Ω bounded in the direction $\overrightarrow{oy}$ then according to the Poincaré inequality A is bounded ie: there is a constant $c' > 0$ such that:

$$\|Au\|_{L^2} \leq c'\|u\|_{H^2(\Omega)}, \text{for all } u \in D\,(A)$$

Since $A \leq -\varepsilon$ for some $\varepsilon > 0$ then $(-A)^{\frac{1}{2}}$ is well defined and bounded, and we admit for the moment the result:

For all $\delta > 0$ exists $K_\delta > 0$ such that:

$$\left\|(-A)^{\frac{1}{2}}u\right\|_{L^2} \leq \delta\|Au\|_{L^2} + K_\delta\|u\|_{L^2}, \in D((-A)^{\frac{1}{2}}) \ (\bullet)$$

As

$$\left\|(-A)^{\frac{1}{2}}u\right\|_{L^2} = \sum_{i=1}^{2} < \nabla u_i, \nabla u_i >, \forall u \in D((-A)^{\frac{1}{2}})$$

Where $D((-A)^{\frac{1}{2}}) = (H_0^1(\Omega))^2$

According to (*) of the proposition 2.2 there is $c > 0$ such that:

$$\|u\|_{H^2(\Omega)} \leq c\|A_S u\|_{L^2(\Omega)}$$

Finally,

$\|T_0 u\|_{L^2} \leq c''((\sum_{i=1}^{2} \langle \nabla u_i, \nabla u_i \rangle)^{\frac{1}{2}} + \|u\|_{L^2}), u \in (H_0^1(\Omega))^2$ and $c'' > 0$

Thus $\|T_0 u\|_{L^2} \leq c''(\delta\|Au\|_{L^2} + K_\delta\|u\|_{L^2}) + \|u\|_{L^2}$

$$\leq c''\delta c'\|u\|_{H^2(\Omega)} + (c''K_\delta + 1)\|u\|_{L^2}$$

$$\leq c''\delta c' c\|A_S u\|_{L^2(\Omega)} + (c''K_\delta + 1)\|u\|_{L^2}$$

<u>Proof of</u> $(\bullet)$:

We know that for all $\epsilon > 0$ we have:

$$\lambda \leq \epsilon\lambda^2 + (4\epsilon)^{-1}, \lambda \geq 0 \qquad (*)$$

And let now A be the operator with the domain of definition $D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^2$ such that $Au = -I\Delta$ for $u = (u_1, u_2) \in D(A)$.

According to proposition 2.2 we have A is selfadjoint and $\geq \delta$ for some $\delta > 0$.

Let $(E_\lambda)_{\lambda \geq \delta}$ be the spectral family associated to the operator A, :

$$A = \int_\delta^\infty \lambda dE_\lambda, \ A^{\frac{1}{2}} = \int_\delta^\infty \lambda^{\frac{1}{2}} dE_\lambda \qquad (**)$$

We use (*) et (**) and according to simple calculus:

$$\| A^{\frac{1}{2}}u \|_{L^2} \leq \epsilon \| Au \|_{L^2}^2 + (4\epsilon)^{-1} \| u \|_{L^2}^2 \in D(A) \ (***)$$

We have obviously:

$$\| Au \|_{L^2}^2 \leq c_0 \| u \|_{H^2}^2, u \in D(A), \text{for one } c_0 > 0$$

In the other hand there various ways to show (for example the quadratic forms theory in [4]) that:

$$D\left(A^{\frac{1}{2}}\right) = H_0^1(\Omega), \ \| A^{\frac{1}{2}}u \|_{L^2} = \textstyle\sum_1^2 \| \nabla u_j \|_{L^2}^2 \ (****)$$

for $u = (u_1, u_2) \in D(A^{\frac{1}{2}}) = H_0^1(\Omega)$.

Take now $u = (u_1, u_2) \in D(A_s)$ and we combine (***), (****) with the proposition 2.2 to obtain:

$$\sum_1^2 \| \nabla u_j \|_{L^2}^2 \leq \epsilon c_0 c \| A_s u \|_{L^2}^2 + (4\epsilon)^{-1} \| u \|_{L^2}^2$$

with c of the proposition 2.2.

Remark 2 According to the corollary 1, T_0 is relatively bounded with respect to A_S then $\mathcal{P}T_0$ too, and according to [3] $A_S + PT_0$ generates a holomorphic semi-group on E.

Then, it possible to define for each fixed $\lambda > 0, \lambda \in \rho(A_S + PT_0)$, fractional powers $B_\lambda^\gamma = (\lambda - A_S - PT_0)^\gamma, (\gamma > 0)$, and we can introduce the spaces of fractional powers X_γ and norms $\|.\|_\gamma$ that all equivalents independently to λ, and defined by :

$$\|f\|_\gamma = \left\|B_\lambda^\gamma f\right\|_{L^2}, f \in X_\gamma = D(B_\lambda^\gamma).$$

According to [3] the non linear operator N verify the next assertion:

If $\gamma \in \left]\frac{3}{4}, 1\right[, v \in X_\gamma$ then $N(v) \in L^2(\Omega)$ and $\|N(v)\|_{L^2} \leq C_\gamma \|v\|_\gamma^2$.

Then the problem (1) became an evolution equation of the sense of Pazy [3].

Definition 2.1 [5]

(U_0, p_0) stable in the sense of Ljapounov of (1) under perturbations which are in $\mathcal{L}^2(\Omega)$, if $v = 0$ is an equilibrium solution of (3) stable in the sense of Ljapounov.

Theorem 1 [5]
1. If $\sigma(B) \subseteq \{\lambda/\mathrm{Re}(\lambda) \leq -\delta\}$ for some $\delta > 0$ then $v = 0$ is stable in the sense of Ljapounov of (3),
2. If $\mathrm{Re}(\lambda) > 0$ for some $\lambda \in \sigma(B)$ then $v = 0$ is unstable in the sense of Ljapounov of (3)

Since the source of the difficulties is the singularity of the Stokes operator then we will focus on it using the same theory. This theory, well established in the case of Schrödinger equations with periodic potentials [4] extends to the stokes operators which occur in Navier-Stokes and related equations, but the corresponding theory is now more involved, see [7] where the three dimensional case (3d) is treated. The Stokes operators which appear in connection with (1), either 2d or 3d, are of the form:

$$A_s U = \nu \mathcal{P} \Delta U. \qquad (4)$$

U is the argument on which the operator acts, while $\mathcal{P}$ is the orthogonal projection onto the space of divergence free fields. Three cases are of interest:
(a) $U \in (H^2(\Omega) \cap H_0^1(\Omega))^2, \mathrm{div}\, U = 0$.
(b) U is periodic in the unbounded space direction.
(c) U is Floquet-periodic in the unbounded space direction.
Case (b) subsumes under case (c) [2]; case (a) is handled in [6]. There is a spectral formulas relation between the cases (a) and (c), well known in case of the Schrödinger equations with periodic potentials. In the 3d-case however, these spectral formulas associated with (2) are more complicated than in the Schrödinger case due to the appearance of singularities [7]. The purpose of this paper is to show that in the 2d-case these singularities are disappear and that the spectral formulas associated with (2) have the same results as in the Schrödinger equations case. To this effect we study first the most important special, ie. $U = 0$. We have to perform estimates similar to those in [7]. In our estimates, which are considerably simpler, singularities do not appear.

How to exploit this fact so as to obtain the mentioned spectral formulas is the outlined in subsequent section.

III. Θ-Periodic function

We fix a period $L > 0$, set $Q_L =]0, L[$ and $Q = Q_L \times]-\frac{1}{2}, \frac{1}{2}[$, for some small $\epsilon > 0$ we put $M_\epsilon =]-\epsilon, 2\pi + \epsilon[$ with $M = [0,2\pi]$ (M_ϵ is a neighbourhood of M) .
And let $\dot{M}_\epsilon$ be M_ϵ minus the numbers 0 and 2π.
We define a Θ-Periodic function (Floquet-periodic function):
For Θ in M_ϵ ; $f \in C_\Theta^p(Q)$ if $f \in C^p(Q)$ and $f(x + jL, y) = e^{ij\Theta} f(x, y), j \in ZZ, (x, y) \in \overline{\Omega}$, with $i^2 = -1$ we define the functional spaces :

$H_\Theta^p(Q)$ is the set of $f \in \mathcal{L}^2(Q)$ such that $\lim_{n \to +\infty} \| f_n - f \|_{H^p} = 0$ for some sequence $(f_n)_{n \in IN} \in C_\Theta^p(Q)$

We also let $\mathcal{L}_g^2$ be the subspace of $\mathcal{L}^2(Q)$ containing the elements f such that $f(x, -y) = f(x, y)\ a.e.$
Likewise with $\mathcal{L}_u^2$ and $f(x, -y) = -f(x, y)\ a.e$
Finaly, we put

$$L^2 = (\mathcal{L}^2)^2, L_g^2 = \mathcal{L}_g^2 \times \mathcal{L}_u^2 \text{ and } \mathcal{L}_u^2 = \mathcal{L}_u^2 \times \mathcal{L}_g^2$$

It's easy to prove that :

$$L^2 = L_g^2 \oplus L_u^2$$

IV. Fourier series

We consider the eigenvalue problem: $y'' + \lambda y = 0$ on $]-\frac{1}{2}, \frac{1}{2}[$ with Neumann resp. Dirichlet boundary conditions.

In the first case we have a complete orthonormal (C.O.N) system in $\mathcal{L}^2(Q)$:
$\varphi_{2k} = (-1)^k \sqrt{2} \cos(2k\pi y)$ for $k \geq 1, \varphi_0 = 1$

$\varphi_{2k+1} = (-1)^k\sqrt{2}\sin\big((2k+1)\pi y\big)$ for $k \geq 0$

$\Lambda_p = p^2\pi^2$ is an eigenvalue associated to φ_p, φ_{2k} is even, φ_{2k+1} odd and morever $\varphi_p(1/2) = \sqrt{2}$ for $p \geq 1$.

And in other case we have a (C.O.N) system given by $\sqrt{\Lambda_p}\psi_p = \varphi'_p$ where $\psi'_p = -\sqrt{\Lambda_p}\varphi_p$ for $p \geq 1$.
Since parity in y will be important we introduce notations:

$\sigma_k = \varphi_{2k+1}, \tau_k = \psi_{2k+1}, \lambda_k = \Lambda_{2k+1}, k \geq 0$, and
$\rho_k = \varphi_{2k}, \pi_k = \psi_{2k}$ for $k \geq 1, \varphi_0 = 1$ and $\mu_k = \lambda_{2k}$
And for $\Theta \in M_\epsilon$ we set: $\hat{\alpha} = (2\pi\alpha + \Theta)L^{-1}, \alpha \in \mathbb{Z}$ and $e_\alpha = e^{i\hat{\alpha}x}$.

We have a characterization of spaces $H^1_{\Theta,0}, H^1_\Theta, H^2_\Theta$ with the Fourier series :

Let $f \in \mathcal{L}^2(Q)$ have Fourier series :

$$f = \sum f_{\alpha,i}\varphi_i = \sum \tilde{f}_{\alpha,i}\psi_i$$

With respect to $\{e_\alpha\varphi_i\}$ resp $\{e_\alpha\psi_i\}$.

Proposition 4.1
(a) $f \in H^1_\Theta$ iff

$$\sum (\hat{\alpha}^2 + \Lambda_i)|f_{\alpha,i}|^2 < \infty$$

(b) $f \in H^1_{\Theta,0}$ iff

$$\sum (\hat{\alpha}^2 + \Lambda_i)|\tilde{f}_{\alpha,i}|^2 < \infty.$$

For the proof see [7].
We have the characterization of space H^2_Θ too :

Proposition 4.2
Let $f \in \mathcal{L}^2(Q)$ satisfy $\sum (\hat{\alpha}^2 + \Lambda_i)^2|f_{\alpha,i}|^2 < \infty$. then $f \in H^2_\Theta$ and :

$$\| f \|^2_{H^2} \leq C\Big(\sum (\hat{\alpha}^2 + \Lambda_i)^2|f_{\alpha,i}|^2\Big)$$

for a C independent of $\Theta \in M_\epsilon$.
Likewise with $\sum (\hat{\alpha}^2 + \Lambda_i)^2|\tilde{f}_{\alpha,i}|^2$.
For the proof see [7].

V. Main theorem

Our aim is to prove:
Theorem 2
(a) There is $C > 0$ as follows:
If $U \in dom(A_s(\Theta)) \cap E^g_\Theta$ and $A_s(\Theta)U = f$ for some $\Theta \in M_\epsilon, f \in E^g_\Theta$ then $U \in (H^2_\Theta)^2$ and

$$\|U\|_{H^2} \leq C\|f\|_{\mathcal{L}^2}$$

(b) Under the conditions $U \in dom(A_s(\Theta)) \cap E^u_\Theta$ or $U \in dom(A_s(\Theta)) \cap E_\Theta$ the assertion (a) holds.

Proposition 5.1
If $f \in H^1_\Theta$ has Fourier series $\sum_{\alpha,j} a_{\alpha,j}e_\alpha\sigma_j$ then $\sum_j |a_{\alpha,j}| \leq \infty$ and $f \in H^1_{\Theta,0}$ iff $\sum_j a_{\alpha,j} = 0, \alpha$ in $\mathbb{Z}$

For the proof of the proposition 5.1 see [7].
We need also the proposition 6.12 used in [7] page 115:
We recall $\lambda_k = \frac{(2k+1)^2}{\pi^2}$:
Proposition 5.2 *There are* Γ_0, Γ_1 *such that for* $s \geq 0$*:*
(i) $\Gamma_0(1+s)^{-3} \leq \sum (\lambda_k + s^2)^{-2} \leq \Gamma_1(1+s)^{-3}$
(ii) $\sum (\lambda_k + s^2)^{-1} \leq \Gamma_1(1+s)^{-1}$
(iii) $\sum \lambda_k^{-1}(\lambda_k + s^2)^{-2} \leq \Gamma_1(1+s)^{-4}$
(iv) $\sum \lambda_k(\lambda_k + s^2)^{-2} \leq \Gamma_1(1+s)^{-1}$

VI. Proof of the main theorem

Since in the first part of the proof factor $\hat{\alpha}^{-1}$ appear which later cancel, it is advantagious to assume first $\Theta \in \dot{M}_\epsilon$.
We take $U = (A, B) \in (H^1_{\Theta,0}) \cap L^2_g$ such that $div\, U = 0$.
We know that $L^2_g = \mathcal{L}^2_g \times \mathcal{L}^2_u$ then $A \in \mathcal{L}^2_g$ and $B \in \mathcal{L}^2_u$ and with characterization of space $H^1_{\Theta,0}$ by Fourier series we have $A = \sum A_{j\alpha}e_\alpha\tau_j, B = \sum B_{j\alpha}e_\alpha\sigma_j$ such that $\sum (\lambda_j + \hat{\alpha}^2)|A_{j\alpha}|^2 < \infty$, likewise for B , the components of $f = (a, b)$ admit expansions too, $a = \sum a_{j\alpha}e_\alpha\tau_j, b = \sum b_{j\alpha}e_\alpha\sigma_j$.

U is a weak solution of $A_s(\Theta)U = f$ for $f \in E_\Theta$ if and only if:

$$\sum_{j=1}^{2} \langle \nabla U_j, \nabla V_j \rangle + \langle f, V \rangle = 0 \qquad (5)$$

for all $V \in (H^1_{\Theta,0})^2$.

As a test vector in (5) we take:

$V = (u_0\tau_0 + u_j\tau_j, w_0\sigma_0 + w_j\sigma_j) \in (H^1_{\Theta,0})^2$ Whereby $div\ V = 0$ thus:

$\sqrt{\lambda_j} w_j = -\partial_x u_j$ and $\sqrt{\lambda_0} w_0 = -\partial_x u_0$.

Here $u_0 \in H^2_\Theta(Q_L)$ arbitrarily fixed.

Like in paper [7] we have $w_0 + w_j = 0$.Then from the divergence condition we deduce that $\frac{1}{\sqrt{\lambda_0}} u_0 + \frac{1}{\sqrt{\lambda_j}} u_j$ it's aconstant Θ-periodic then $u_j = -\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_0}} u_0$.

By exploiting the arbitrariness of u_0, ψ we reach certain equations for the Fourier coefficients $A_{j,\alpha}, B_{j,\alpha}, a_{j,\alpha}, b_{j,\alpha}$.

We note :

$$\hat{\lambda}_j = \lambda_j + \hat{\alpha}^2, j \geq 0, \alpha \in \mathbb{Z}$$
$$(A)_j(\alpha) = \hat{\lambda}_j A_{j,\alpha} - a_{j,\alpha}, j \geq 0, \alpha \in \mathbb{Z}$$
$$(B)_j(\alpha) = \hat{\lambda}_j B_{j,\alpha} - b_{j,\alpha}, j \geq 0, \alpha \in \mathbb{Z}$$

We obtain:

$$-\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_0}}(A)_j(\alpha) + (A)_0(\alpha) - \frac{i\hat{\alpha}}{\sqrt{\lambda_0}}(B)_j(\alpha) + \frac{i\hat{\alpha}}{\sqrt{\lambda_0}}(B)_0(\alpha) = 0\ , j \geq 0 \qquad (6)$$

And from the divergence condition for u, f we get:

$$(B)_j(\alpha) = -\frac{i\hat{\alpha}}{\sqrt{\lambda_j}}(A)_j(\alpha), j \geq 0 \qquad (7)$$

From the condition $\Theta \in \dot{M}_\epsilon$ we get $\hat{\alpha} \neq 0$ then:

$$(A)_j(\alpha) = \frac{i\sqrt{\lambda_j}}{\hat{\alpha}}(B)_j(\alpha), j \geq 0 \qquad (8)$$

So according to (6) and (8) we have:

$$\hat{\lambda}_j (B)_j(\alpha) = \hat{\lambda}_0 (B)_0(\alpha), j \geq 0 \qquad (9)$$

And by using proposition 5.1 we have $\sum_j B_{j\alpha} = 0$ then:

$$\begin{cases} B_{0,\alpha} = k(\hat{\lambda}_0 \sum_{j\geq 1} (\hat{\lambda}_j)^{-2} b_{0,\alpha} - \sum_{j\geq 1} (\hat{\lambda}_j)^{-1} b_{j,\alpha}) \\ k = (1 + (\hat{\lambda}_0)^2 \sum_{j\geq 1} (\hat{\lambda}_j)^{-2})^{-1} = k(\alpha) \end{cases} \qquad (10)$$

Having $B_{0,\alpha}$, we can express $B_{j,\alpha}, j \geq 1$ via (9) and then $A_{j,\alpha}, j \geq 0$ via (8).

Then (6) became:

$$\frac{\hat{\lambda}_j}{\sqrt{\lambda_j}}(A)_j(\alpha) = \frac{\hat{\lambda}_0}{\sqrt{\lambda_0}}(A)_0(\alpha) \qquad (11)$$

And (8) give us (for $j = 0$):

$$(A)_0(\alpha) = \frac{i\sqrt{\lambda_0}}{\hat{\alpha}}(B)_0(\alpha)$$

So

$$(A)_j(\alpha) = \frac{i\sqrt{\lambda_j}\hat{\lambda}_0}{\hat{\alpha}\hat{\lambda}_j}(B)_0(\alpha) \qquad (12)$$

And from (10) we deduce:

$$(B)_0(\alpha) = -k(b_{0,\alpha} + \hat{\lambda}_0 \sum_{j\geq 1} (\hat{\lambda}_j)^{-1} b_{j,\alpha}) \qquad (13)$$

By divergence condition we replace $b_{j,\alpha}$ by $a_{j,\alpha}$ in (13).

If we replace $(B)_0(\alpha)$ in (12) by it's value we obtain:

$$(A)_j(\alpha) = \frac{-\sqrt{\lambda_j}\hat{\lambda}_0 k}{\hat{\lambda}_j}\Big(\frac{1}{\sqrt{\lambda_0}} a_{0,\alpha} + \hat{\lambda}_0 \sum_{s\geq 1} \left(\lambda_s^{\frac{1}{2}} (\hat{\lambda}_s)^{-1} a_{s,\alpha}\right) \qquad (14)$$

As can be read off from (14), the expression for $(A)_j(\alpha)$ does not contain any factor $\hat{\alpha}^{-1}$, that is no singularity, we may therefore assume from now on that $\Theta \in M_\epsilon$. (6)

By (14) we have:

$$(A)_j(\alpha) = I_j + II_j$$

where (7)

$$I_j = \frac{-\sqrt{\lambda_j}\hat{\lambda}_0 k}{\hat{\lambda}_j} \hat{\lambda}_0 \sum_{j\geq 1} (\lambda_j^{1/2} \hat{\lambda}_j)^{-1} a_{j,\alpha}$$

and

$$II_j = \frac{-\sqrt{\lambda_j}\hat{\lambda}_0 k}{\hat{\lambda}_j \sqrt{\lambda_0}} a_{0,\alpha}$$

We note that by proposition 5.2 (i) a Γ_2 is found such that,

$k \leq \Gamma_2 (1 + s)^{-1}, (s = |\hat{\alpha}|)$, then:

$$|I_j|^2 \leq$$
$$\frac{\lambda_j \hat{\lambda}_0{}^2 k^2}{\hat{\lambda}_j{}^2} \left(\sum_{s\geq 1} (\hat{\lambda}_s)^{-2} (\hat{\lambda}_0)^2 (\lambda_s)^{-1}\right)\left(\sum_{s\geq 1} |a_{s,\alpha}|^2\right) \quad (15)$$
$$\leq$$
$$\frac{\Gamma_2^2(1+s)^{-2}(\lambda_0+s)^2\lambda_j}{(\lambda_j+s^2)^2} \left(\sum_{s\geq 1} \lambda_s^{-1}\right)\left(\sum_{s\geq 1} |a_{s,\alpha}|^2\right) \quad (16)$$
$$\leq \frac{C'}{\lambda_j} \sum_{s\geq 1} |a_{s,\alpha}|^2 \quad (17)$$

So

$$\sum_{\alpha}\sum_{j\geq 1} |I_j|^2 \leq C \sum_{\alpha}\sum_{s\geq 1} |a_{s,\alpha}|^2$$

and for II_j we have: $|II_j|^2 = \frac{\lambda_j \hat{\lambda}_0{}^2 k^2}{\hat{\lambda}_j{}^2 \lambda_0} |a_{0,\alpha}|^2$

then:

$$|II_j|^2 \leq \frac{\lambda_j (\lambda_0 + s^2)^2 \Gamma_2^2 (1+s)^{-2}}{(\lambda_j + s^2)^2 \lambda_0} |a_{0,\alpha}|^2$$

Then

$$\sum_{j\geq 1} |II_j|^2 \leq C'(1+s)^{-2} |a_{0,\alpha}|^2$$

Therefore

$$\sum_{\alpha}\sum_{j\geq 1} |II_j|^2 \leq C_1 \sum_{\alpha} |a_{0,\alpha}|^2$$

We still have to look at $(A)_0(\alpha)$. We recall (14) for $j = 0$ and we can estimate $k(\alpha)$ by Proposition 5.1

And for $(B)_j(\alpha)$: By (7) and (12) we can deduce by using Proposition 5.1 that there is a Θ −independent C_2 such that :

$$\sum_{\alpha}\sum_{j} \left|(B)_j(\alpha)\right|^2 \leq C_2 \sum_{\alpha}\sum_{j} \left|b_{j,\alpha}\right|^2$$

The proof of (b) is very similar.

Conclusion:

$$\| U \|_{H^2} \leq C \| f \|_{\mathcal{L}^2}.$$

VII. Comments

As indicated, due to the fact that the singularity $\Theta = 0$ resp. $\Theta = 2\pi$ drops out in the computations presented in the previous section, the spectral theory, carried out for dimension $d = 3$ in [7] is simplified considerably. Partly for this reason and partly for reasons of space, we consent us to describe briefly the final result which emerges from this simplification. In order to describe the way in which the spectral formula in [7] simplifies, we recall the objects which appear in it. Following sect. 1, 3. we have the Θ-periodic Sobolev spaces $H^p_\Theta(Q), H^1_{\Theta,0}(Q), \Theta \in]-\varepsilon, 2\pi+\varepsilon[$, the orthogonal projection $\mathcal{P}_\Theta$ from $\mathcal{L}^2(Q)^2$ onto E_Θ, with E_Θ the $\mathcal{L}^2$-closure of the set of $f \in H^1_\Theta(Q) \times H^1_{\Theta,0}(Q)$ such that $div\, f = 0$. The Θ -periodic Stokes operator $A_S(\Theta)$ are now defined as follows:

$$\begin{cases} f \in dom(A_S\Theta))\ iff\ f \in (H^2_\Theta(Q) \cap H^1_{\Theta,0}(Q))^2 \\ and\ div\, f = 0, and\ for\ such\ f, A_S(\Theta) f = \nu \mathcal{P}_\Theta \Delta f. \end{cases}$$

$$\begin{cases} f \in dom(A_S\Theta))\ iff\ f \in (H^2_\Theta(Q) \cap H^1_{\Theta,0}(Q))^2 \\ and\ div\, f = 0, and\ for\ such\ f, A_S(\Theta) f = \nu \mathcal{P}_\Theta \Delta f. \end{cases}$$

Next, we recall that, as stressed in the introduction, we are given a smooth velocity field $v = (v_1, v_2)$ on $IR \times [\frac{1}{2}, \frac{1}{2}]$ which is L-periodic in the unbounded variable x, which gives rise to an operator T which acts on elements $u = (u_1, u_2) \in dom(A_S(\Theta))$ according to

$$Tu = -\left(v_1 \partial_x u_1 + v_2 \partial_y u_1, v_1 \partial_x u_2 + v_2 \partial_y u_2\right) \quad (18)$$

We briefly digress on the periodic case which arises for $\Theta = 0$ and $\Theta = 2\pi$. In accordance with [7] we stress this case by the label 'per' rather than by $\Theta = 0$ or $\Theta = 2\pi$. Thus $A_S(per) = A_S(0) = A_S(2\pi)$, $H^p_{per}(Q) = H^p_0(Q) = H^p_{2\pi}(Q)$ etc. In order to achieve that the spectral formulas below are valid, we have to define $E_{per}, A_S(per), \mathcal{P}_{per}$ as follows:

E_{per} is the $\mathcal{L}^2$ − closure of all vector fields

$$v = (f, h) in H^1_{per}(Q) \times H^1_{per,0}(Q)$$

Such that div $f = 0$ and $\int_Q f\, dxdy = 0$.

$v = (f, h)$ is in dom $(A_S(per))$ if $v \in (H^2_{per}(Q) \cap H^1_{per,0}(Q))^2$,

div $v = 0$ and $\int_Q f\, dxdy = 0$;

for such v we set $A_S(per)v = \nu\mathcal{P}_{\text{per}}\Delta v$, $where\ \mathcal{P}_{per}$ is the orthogonal projection from $\mathcal{L}^2(Q)^2$ onto E_{per}. With this definition, $A_S(\text{per})$ is selfadjoint on E_{per}. Finally, we need corresponding objects defined on the whole strip $\Omega = IR\ \times]-\frac{1}{2},\frac{1}{2}[$. Thus E is the $\mathcal{L}^2$ −closure of $f \in H^1(\Omega) \times H_0^1(\Omega)$ such that div $f = 0$,

$f \in \text{d}om(A_S)\ iff\ f \in (H^2(\Omega) \cap$
$H_0^1(\Omega))^2\ and\ div\ f = 0$,

and for such f we set $A_S f = \nu\mathcal{P}\Delta f$.

For elements $f \in \text{d}om(A_S)$, the operator T acts again via (18). Under these conditions, the operators

$$G = A_S + \mathcal{P}T,$$
$$G_\Theta = A_S(\Theta) + \mathcal{P}_\Theta T,$$
$$G_{\text{per}} = A_S(\text{per}) + \mathcal{P}_{\text{per}} T$$

all become holomorphic semigroup generators on $E, E_\Theta, E_{\text{per}}$ respectively. The spectral formulas, announced above now are:

$(22)_1\ \sigma(A_S + \mathcal{P}T) = \overline{\left(\bigcup_{\Theta\in[0,2\pi]} (A_S(\Theta) + \mathcal{P}_\Theta T)\right)}$

$$(22)_2 \qquad \sigma(A_S + \mathcal{P}T) = \bigcup_{\Theta\in[0,2\pi]} (A_S(\Theta) + \mathcal{P}_\Theta T).$$

These formulas correspond to formulas (*), (**) in [7], page. 169. While $(22)_1$ looks the same as (*) in [7], $(22)_2$ is definitely simpler; it implies in particular that if $\lambda \in \sigma(A_S(\text{per}) + \mathcal{P}_{\text{per}}T)$ then $\lambda \in \sigma(A_S + \mathcal{P}T)$, a statement which cannot be asserted in dimension $d = 3$ as can be seen from formula (**) in [7]. The proof of $(22)_2$ is based on the computations in the present section 5, which entails that the singularities which arise in dimension $d = 3$ in [7], drop out.

The detailed verification of this claim is treated by a careful examination of the arguments in [7], a task within the scope of this paper.

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