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Derivation and Implementation of a Linear Multistep Numerical Scheme of Order 12

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ABSTRACT

A great many physical occurrences give rise to problems that often result in differential equations. When we solve a differential equation, we are in effect solving the physical problem it represents. Traditionally, solutions to differential equations were derived using analytical or exact methods. These solutions are often useful as they provide excellent insight into the behavior of some systems. However, certain differential equations are very difficult to solve by any means other than an approximate solution by the application of numerical methods. These methods can be classified into two thus: One-step and multistep methods. However, in this work, our focus is on a class of multistep methods known as the Linear Multi-step Methods. We thus use Taylor series expansion to derive a linear multistep method of order 12. The method is tested for consistency and zero-stability in order to establish its convergence. Also, provided are some examples of problems solved using the new scheme, and the results compared with exact solution.

Keywords : Linear Multi-step, Ordinary differential equation, Numerical Scheme, Taylor Series.

I. INTRODUCTION

Initial Value Problem for First-Order Ordinary Differential Equation

A first-order differential equation $y' = f(t,y)$, may posses an infinite number of solution. For instance, the function $y(x) = Ce^{\lambda x}$is, for any value of the constant C, a solution of the differential equation $y' = \lambda y$, where λ is given constant. We can pick out any particular solution by prescribing an *initial condition*, $y(a) = \eta$. For these example, the particular solution satisfying this initial condition is easily found to be $y(x) = \eta e^{\lambda(x-q)}$. We say that the differential equation together with an initial condition constitutes an *initial value problem*,

$$y' = f(t,y), y(a) = \eta \tag{1}$$

The following theorem, whose proof is as documented by Henrici (1996) states conditions on $f(x,y)$ which guarantee the existence of a unique solution of the initial value problem (1)

Theorem 1.1 : Let $f(x,y)$ be defined and continuous for all points (x,y) in the region D defined by $a \le x \le b, -\infty < y < \infty, a$ and b finite and let there exist a constant L such that for every x, y, y^* such that (x,y) and (x,y^*) are both in D,

$$|f(x,y) - f(x,y^*)| \le L|y - y^*|. \tag{2}$$

Then, if η is any given number, there exists a unique solution $y(x)$ of the initial value problem (1), where

$y(x)$ is continuous and differentiable for all (x, y) in D.

The requirement (2) is known as a Lipschitz condition, and the constant L as a Lipschitz constant. This condition may be thought of as being intermediate between differential and continuity, in the sense that

$\Longrightarrow f(x, y)$ continously differentiable with respect to y for all (x, y) in D

$\Longrightarrow f(x, y)$satisfies a Lipschitz condition with respect to y for all (x, y) in D

$\Longrightarrow f(x, y)$ continuous with respect to y for all (x, y) in D.

In particular, if (x, y) possesses a continuous derivative with respect to y for all (x, y) in D, then by the mean value theorem,

$$f(x, y) - f(x, y^*) = \frac{\partial f(x, \bar{y})}{\partial y}(y - y^*),$$

where $\bar{y}$ is a point in the interior of the interval whose end-points are y and y^*, and (x, y) and (x, y^*) are both in D. clearly, (2) is then satisfied if we choose

$$L = \operatorname*{Sup}_{(x,y)\in D} \left| \frac{\partial(x, y)}{\partial y} \right| \qquad (3)$$

We consider the initial value problem (IVP) for a single first-order differential equation

$$y' = f(t, y), y(t_0) = y_0, t > t_0 \qquad (4)$$

We seek a solution in the range $a \le x \le b$, where a and b are finite, and we assume that f satisfies the conditions stated in theorem 1.1 which guarantee that the problem has a unique continuously differentiable solution, which we indicate by $y(x)$. Considering the sequence of points $\{x_n\}$ defined by $x_n = a + nh, n = 0,1,2, \dots$ The parameter h, which will always be regarded as constant, except where otherwise indicated, is called the *steplength*. An essential property of the majority of computational methods for the solution of (4) is that of discretizationl that is, we seek an appropriate solution, not on the continuous interval $a \le x \le b$, but on the discrete point set $\{x_n | n = 0,1, \dots, (b - a)/h\}$. Let y_n be an approximation to the theoretical solution at x_n, that is, to $y(x_n)$, and let $f_n \equiv f(x_n, y_n)$. If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, j = 0,1, \dots, k$, we call it a linear multistep method of stepnumber k, or a linear $k - step$ method. The general linear multistep method may this be written

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} * \qquad (5)$$

where α_j and β_j are constants; we assume $\alpha_j \neq 0$ and that not both α_0 and β_0 are zero. Since (5) can be multiplied on both sides by the same constant without altering the relationship, the coefficients α_j and $\beta_j$are arbitrary to the extent of a constant multiplier. We say that the method is explicit if $\beta_k = 0$, $\alpha_j = 1$ and implicit if $\beta_k \neq 0$ [1].

Thus the problem of determining the solution $y(x)$ of the, in general, non-linear initial value problem (1) is replaced by that of the finding a sequence $\{y_n\}$ which satisfies he difference equation (2). Note that, since $f_n (= f(x_n, y_n))$ is, in general, a non-linear function of y_n, (2) is a non-linear difference equation. Such equations are no easier to handle theoretically than are non-linear differential equations, but they have the practical advantage of permitting us to compute the sequence $\{y_n\}$ numerically.

In order to do this, we first supply a set of starting values, $y_0, y_1, \dots, y_{k-1}$. In the case of a one-stop method, only one such value, y_0, is needed, and we normally choose $y_0 = \eta$.

Convergence is a minimal property which any acceptable linear multistep method must possess. Thus, the necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable [1].

Consistency demands that,

$$\rho(1) = 0 \; and \; \rho'(1) = \sigma(1) \qquad (6)$$

where,

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \qquad (7)$$

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \qquad (8)$$

are called the first and second characteristic polynomials of (5) respectively.

The LMM (2) is said to be zero-stable if no root of (7) has modulus greater than one, and if every root with modulus one is simple [1].

Convergence

A basic property which shall be demanded of an acceptable linear multi-step method is the situation $\{y_n\}$ generated by the method converges, in some sense, to the theoretical solution $y(x)$ as the steplength h tends to zero. In converting this intuitive concept into a precise definition, the following points must be kept in mind.

i. It is inappropriate to consider n as remaining fixed while $h \longrightarrow 0$. For example, consider a fixed point $x = x^*$, and let the initial choice of steolength h_0 be such that $x^* = a + 2h_0$. We are led to the idea of 'fixed station convergence' by which we mean convergence in the limit as $h \longrightarrow 0, n \longrightarrow \infty, nh = x - a$ remaining fixed. Such a limit will be written

$$\lim_{\substack{h\longrightarrow 0\\ nh=x-a}}$$

ii. The definition must take account of the additional starting values $y_1, y_2, \dots, y_{k-1}$, which must be supplied when $k \geq 2$.

iii. If the term 'convergent' is to be applied to the method (2), then the convergence property must hold for all initial value problems (1) subject to the hypothesis of theorem 1.1.

Definition: The linear multistep method (2) is said to be convergent if, for all initial value problems (1) subject to the hypotheses of theorem 1.1, we have that

$$\lim_{\substack{h\longrightarrow 0\\ nh=x-a}} y_n = y(x_n)$$

Holds for all $x \in [a, b]$, *and for all solutions* $\{y_n\}$ *for the difference equation (2) satisfying starting conditions* $y_\mu = \eta_\mu(h)$ *for which* $\lim_{h\longrightarrow 0} \eta_\mu(h) = \eta, \mu = 0,1,2, \dots, k-1.$

It should be observed that this definition is very generous in the conditions it imposes on the starting values $y_\mu, \mu = 0,1, \dots, k-1$. It does not demand that these be exact solutions of the initial value problem (1) at the appropriate values for x, but only that, regarded as function of h, they all tend to the given initial value η as $h \longrightarrow 0$. Note that it is not even demanded that $y_0 = \eta$, although this is almost invariably the choice we make for y_0 in practice.

Order and error constant – Derivation of the LMM of Order 12

In this section we are in effect formalizing the Taylor series method for the derivation of linear multistep methods. With the linear multistep method (2), we associate the linear difference operator $\mathcal{L}$ defined by:

$$\mathcal{L}[y(x); h] = \sum_{j=0}^{k} [\alpha_j y(x+jh) - h\,\beta_j y'(x+jh)] \qquad (9)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a\,, b]$.

The reason for introducing this operator is that, by allowing it to operate on an arbitrary test function $y(x)$, which we may assume to have as many higher derivatives as we require, w can formally define the order of accuracy of the operator and of the associated linear multistep method, without invoking the solution of the initial value problem (1) which as we have already observed, may possess only a first derivative.

Suppose we choose to expand $y(x + jh)$ and its derivative $y'(x + jh)$ as Taylor series about x+rh; where r need not necessarily be an integer and collecting terms in (9), we obtain:

$$\mathcal{L}[y(x);h] = D_0 y(x+rh) + D_1 h y'(x+rh) + D_2 h^2 y''(x+rh) + \cdots + D_q h^q y^q(x+rh) \quad (10)$$

where the D_q are constants.

Definition: The difference operator (9) and the associated linear multistep method (2) are said to be of ***order*** *q if, in (10),* $D_0 = D_1 = \cdots = D_q = 0.\ D_{q+1} \neq 0.$

A simple calculation yields the following formulae for the constants D_q expressed in terms of α_j, β_j are:

$$\begin{aligned} D_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \\ D_1 &= -r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + \cdots + (k-r)\alpha_k \\ &\quad -(\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ D_q &= \frac{1}{q!}[(-r)^q\alpha_0 + (1-r)^q\alpha_1 + (2-r)^q\alpha_2 + \cdots + (k-r)^q\alpha_k] \\ &\quad - \frac{1}{(q-1)!}[(-r)^{q-1}\beta_0 + (1-r)^{q-1}\beta_1 + (2-r)^{q-1}\beta_2 + \cdots + (k-r)^{q-1}\beta_k] \\ q &= 2, 3, \ldots \end{aligned} \quad (8)$$

These formulae can be used to derive a linear multistep method of given structure and maximal order.

In this research work, we wish to derive an optimal 8-step method. Therefore, all the roots of the first characteristic polynomial $\rho(\xi)$ must be on the unit circle. We know that $\rho(\xi)$ is a polynomial of degree 8. Hence, by consistency, it has one real root at +1 and another real root at -1. The six remaining roots must be complex. Hence we have:

$\xi_1 = +1, \xi_2 = -1, \xi_3 = e^{i\theta 1}, \xi_4 = e^{-i\theta 1}, \xi_5 = e^{i\theta 2}, \xi_6 = e^{-i\theta 2}, \xi_7 = e^{i\theta 3}, \xi_8 = e^{-i\theta 3}$

Hence:

$\rho(\xi) = (\xi_1 - 1)(\xi_2 + 1)\big(\xi_3 - e^{i\theta 1}\big)\big(\xi_4 - e^{-i\theta 1}\big)\big(\xi_5 - e^{i\theta 2}\big)(\xi_6 - e^{-i\theta 2})(\xi_7 - e^{i\theta 3})(\xi_8 - e^{-i\theta 3})$

$\alpha_8 = +1, \alpha_7 = -3(a+b), \alpha_6 = -2(a+b)\ \alpha_5 = (4ab+1), \alpha_4 = 0, \alpha_3 = -(4ab+1), \alpha_2 = 2(a+b)), \alpha_1 = 3(a+b), \alpha_0 = -1$

We require the method to have order 12. We now state the order requirement in terms of the coefficients D_q.

From (8) we have the following:

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$$

$$D_1 = -r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + (3-r)\alpha_3 + (4-r)\alpha_4 + (5-r)\alpha_5 + (6-r)\alpha_6 + (7-r)\alpha_7 + (8-r)\alpha_8 - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8)$$

$$D_2 = \frac{1}{2!}[(-r)^2\alpha_0 + (1-r)^2\alpha_1 + (2-r)^2\alpha_2 + (3-r)^2\alpha_3 + (4-r)^2\alpha_4 + (5-r)^2\alpha_5 + (6-r)^2\alpha_6 + (7-r)^2\alpha_7 + (8-r)^2\alpha_8] - [-r\beta_0 + (1-r)\beta_1 + (2-r)\beta_2 + (3-r)\beta_3 + (4-r)\beta_4 + (5-r)\beta_5 + (6-r)\beta_6 + (7-r)\beta_7 + (8-r)\beta_8]$$

$$D_3 = \frac{1}{3!}[(-r)^3\alpha_0 + (1-r)^3\alpha_1 + (2-r)^3\alpha_2 + (3-r)^3\alpha_3 + (4-r)^3\alpha_4 + (5-r)^3\alpha_5 + (6-r)^3\alpha_6 + (7-r)^3\alpha_7 + (8-r)^3\alpha_8] - \frac{1}{2!}[(-r)^2\beta_0 + (1-r)^2\beta_1 + (2-r)^2\beta_2 + (3-r)^2\beta_3 + (4-r)^2\beta_4 + (5-r)^2\beta_5 + (6-r)^2\beta_6 + (7-r)^2\beta_7 + (8-r)^2\beta_8]$$

$$D_4 = \frac{1}{4!}[(-r)^4\alpha_0 + (1-r)^4\alpha_1 + (2-r)^4\alpha_2 + (3-r)^4\alpha_3 + (4-r)^4\alpha_4 + (5-r)^4\alpha_5 + (6-r)^4\alpha_6 + (7-r)^4\alpha_7 + (8-r)^4\alpha_8] - \frac{1}{2!}[(-r)^3\beta_0 + (1-r)^3\beta_1 + (2-r)^3\beta_2 + (3-r)^3\beta_3 + (4-r)^3\beta_4 + (5-r)^3\beta_5 + (6-r)^3\beta_6 + (7-r)^3\beta_7 + (8-r)^3\beta_8]$$

$$D_5 = \frac{1}{5!}[(-r)^5\alpha_0 + (1-r)^5\alpha_1 + (2-r)^5\alpha_2 + (3-r)^5\alpha_3 + (4-r)^5\alpha_4 + (5-r)^5\alpha_5 + (6-r)^5\alpha_6 + (7-r)^5\alpha_7 + (8-r)^5\alpha_8] - \frac{1}{2!}[(-r)^4\beta_0 + (1-r)^4\beta_1 + (2-r)^4\beta_2 + (3-r)^4\beta_3 + (4-r)^4\beta_4 + (5-r)^4\beta_5 + (6-r)^4\beta_6 + (7-r)^4\beta_7 + (8-r)^4\beta_8]$$

$$D_6 = \frac{1}{6!}[(-r)^6\alpha_0 + (1-r)^6\alpha_1 + (2-r)^6\alpha_2 + (3-r)^6\alpha_3 + (4-r)^6\alpha_4 + (5-r)^6\alpha_5 + (6-r)^6\alpha_6 + (7-r)^6\alpha_7 + (8-r)^6\alpha_8] - \frac{1}{2!}[(-r)^5\beta_0 + (1-r)^5\beta_1 + (2-r)^5\beta_2 + (3-r)^5\beta_3 + (4-r)^5\beta_4 + (5-r)^5\beta_5 + (6-r)^5\beta_6 + (7-r)^5\beta_7 + (8-r)^5\beta_8]$$

$$D_7 = \frac{1}{7!}[(-r)^7\alpha_0 + (1-r)^7\alpha_1 + (2-r)^7\alpha_2 + (3-r)^7\alpha_3 + (4-r)^7\alpha_4 + (5-r)^7\alpha_5 + (6-r)^7\alpha_6 + (7-r)^7\alpha_7 + (8-r)^7\alpha_8] - \frac{1}{2!}[(-r)^6\beta_0 + (1-r)^6\beta_1 + (2-r)^6\beta_2 + (3-r)^6\beta_3 + (4-r)^6\beta_4 + (5-r)^6\beta_5 + (6-r)^6\beta_6 + (7-r)^6\beta_7 + (8-r)^6\beta_8]$$

$$D_8 = \frac{1}{8!}[(-r)^8\alpha_0 + (1-r)^8\alpha_1 + (2-r)^8\alpha_2 + (3-r)^8\alpha_3 + (4-r)^8\alpha_4 + (5-r)^8\alpha_5 + (6-r)^8\alpha_6 + (7-r)^8\alpha_7 + (8-r)^8\alpha_8] - \frac{1}{7!}[(-r)^7\beta_0 + (1-r)^7\beta_1 + (2-r)^7\beta_2 + (3-r)^7\beta_3 + (4-r)^7\beta_4 + (5-r)^7\beta_5 + (6-r)^7\beta_6 + (7-r)^7\beta_7 + (8-r)^7\beta_8]$$

$$D_9 = \frac{1}{9!}[(-r)^9\alpha_0 + (1-r)^9\alpha_1 + (2-r)^9\alpha_2 + (3-r)^9\alpha_3 + (4-r)^9\alpha_4 + (5-r)^9\alpha_5 + (6-r)^9\alpha_6 + (7-r)^9\alpha_7 + (8-r)^9\alpha_8] - \frac{1}{8!}[(-r)^8\beta_0 + (1-r)^8\beta_1 + (2-r)^8\beta_2 + (3-r)^8\beta_3 + (4-r)^8\beta_4 + (5-r)^8\beta_5 + (6-r)^8\beta_6 + (7-r)^8\beta_7 + (8-r)^8\beta_8]$$

$$D_{10} = \frac{1}{10!}[(-r)^{10}\alpha_0 + (1-r)^{10}\alpha_1 + (2-r)^{10}\alpha_2 + (3-r)^{10}\alpha_3 + (4-r)^{10}\alpha_4 + (5-r)^{10}\alpha_5 + (6-r)^{10}\alpha_6 + (7-r)^{10}\alpha_7 + (8-r)^{10}\alpha_8] - \frac{1}{9}[(-r)^9\beta_0 + (1-r)^9\beta_1 + (2-r)^9\beta_2 + (3-r)^9\beta_3 + (4-r)^9\beta_4 + (5-r)^9\beta_5 + (6-r)^9\beta_6 + (7-r)^9\beta_7 + (8-r)^9\beta_8]$$

$$D_{11} = \frac{1}{11!}[(-r)^{11}\alpha_0 + (1-r)^{11}\alpha_1 + (2-r)^{11}\alpha_2 + (3-r)^{11}\alpha_3 + (4-r)^{11}\alpha_4 + (5-r)^{11}\alpha_5 + (6-r)^{11}\alpha_6 + (7-r)^{11}\alpha_7 + (8-r)^{11}\alpha_8] - \frac{1}{9}[(-r)^{10}\beta_0 + (1-r)^{10}\beta_1 + (2-r)^{10}\beta_2 + (3-r)^{10}\beta_3 + (4-r)^{10}\beta_4 + (5-r)^{10}\beta_5 + (6-r)^{10}\beta_6 + (7-r)^{10}\beta_7 + (8-r)^{10}\beta_8]$$

$$D_{12} = \frac{1}{12!}[(-r)^{12}\alpha_0 + (1-r)^{12} + (2-r)^{12}\alpha_2 + (3-r)^{12}\alpha_3 + (4-r)^{12}\alpha_4 + (5-r)^{12}\alpha_5 + (6-r)^{12}\alpha_6 + (7-r)^{12}\alpha_7 + (8-r)^{12}\alpha_8] - \frac{1}{9}[(-r)^{11}\beta_0 + (1-r)^{11}\beta_1 + (2-r)^{11}\beta_2 + (3-r)^{11}\beta_3 + (4-r)^{11}\beta_4 + (5-r)^{11}\beta_5 + (6-r)^{11}\beta_6 + (7-r)^{11}\beta_7 + (8-r)^{11}\beta_8]$$

Arbitrarily Setting $r = 3$ and $D_q = 0$, *we solve for* $q = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

we have:

$$D_2 = \frac{1}{2!}[3^2\alpha_0 + 2^2\alpha_1 + \alpha_2 + \alpha_4 + 2^2\alpha_5 + 3^2\alpha_6 + 4^2\alpha_7 + 5^2\alpha_8] - [-3\beta_0 - 2\beta_1 - \beta_2 + \beta_4 + 2\beta_5 + 3\beta_6 + 4\beta_7 + 5\beta_8] = 0$$

$$D_3 = \frac{1}{3!}[-3^3\alpha_0 - 2^3\alpha_1 - \alpha_2 + \alpha_4 + 2^3\alpha_5 + 3^3\alpha_6 + 4^3\alpha_7 + 5^3\alpha_8] - \frac{1}{2!}[3^2\beta_0 + 2^2\beta_1 + \beta_2 + \beta_4 + 2^2\beta_5 + 3^2\beta_6 + 4^2\beta_7 + 5^2\beta_8] = 0$$

$$D_4 = \frac{1}{4!}[3^4\alpha_0 + 2^4\alpha_1 + \alpha_2 + \alpha_4 + 2^4\alpha_5 + 3^4\alpha_6 + 4^4\alpha_7 + 5^4\alpha_8] - \frac{1}{3!}[-3^3\beta_0 - 2^3\beta_1 - \beta_2 + \beta_4 + 2^3\beta_5 + 3^3\beta_6 + 4^3\beta_7 + 5^3\beta_8] = 0$$

$$D_5 = \frac{1}{5!}[-3^5\alpha_0 - 2^5\alpha_1 - \alpha_2 + \alpha_4 + 2^5\alpha_5 + 3^5\alpha_6 + 4^5\alpha_7 + 5^5\alpha_8] - \frac{1}{4!}[3^4\beta_0 + 2^4\beta_1 + \beta_2 + \beta_4 + 2^4\beta_5 + 3^4\beta_6 + 4^4\beta_7 + 5^4\beta_8] = 0$$

$$D_6 = \frac{1}{6!}[3^6\alpha_0 + 2^6\alpha_1 + \alpha_2 + \alpha_4 + 2^6\alpha_5 + 3^6\alpha_6 + 4^6\alpha_7 + 5^6\alpha_8] - \frac{1}{5!}[-3^5\beta_0 - 2^5\beta_1 - \beta_2 + \beta_4 + 2^5\beta_5 + 3^5\beta_6 + 4^5\beta_7 + 5^5\beta_8] = 0$$

$$D_7 = \frac{1}{7!}[-3^7\alpha_0 - 2^7\alpha_1 - \alpha_2 + \alpha_4 + 2^7\alpha_5 + 3^7\alpha_6 + 4^7\alpha_7 + 5^7\alpha_8] - \frac{1}{6!}[3^6\beta_0 + 2^6\beta_1 + \beta_2 + \beta_4 + 2^6\beta_5 + 3^6\beta_6 + 4^6\beta_7 + 5^6\beta_8] = 0$$

$$D_8 = \frac{1}{8!}[3^8\alpha_0 + 2^8\alpha_1 + \alpha_2 + \alpha_4 + 2^8\alpha_5 + 3^8\alpha_6 + 4^8\alpha_7 + 5^8\alpha_8] - \frac{1}{7!}[-3^7\beta_0 - 2^7\beta_1 - \beta_2 + \beta_4 + 2^7\beta_5 + 3^7\beta_6 + 4^7\beta_7 + 5^7\beta_8] = 0$$

$$D_9 = \frac{1}{9!}[-3^9\alpha_0 - 2^9\alpha_1 - \alpha_2 + \alpha_4 + 2^9\alpha_5 + 3^9\alpha_6 + 4^9\alpha_7 + 5^9\alpha_8] - \frac{1}{8!}[3^8\beta_0 + 2^8\beta_1 + \beta_2 + \beta_4 + 2^8\beta_5 + 3^8\beta_6 + 4^8\beta_7 + 5^8\beta_8] = 0$$

$$D_{10} = \frac{1}{10!}[3^{10}\alpha_0 + 2^{10}\alpha_1 + \alpha_2 + \alpha_4 + 2^{10}\alpha_5 + 3^{10}\alpha_6 + 4^{10}\alpha_7 + 5^{10}\alpha_8] - \frac{1}{9!}[-3^9\beta_0 - 2^9\beta_1 - \beta_2 + \beta_4 + 2^9\beta_5 + 3^9\beta_6 + 4^9\beta_7 + 5^9\beta_8] = 0$$

$$D_{11} = \frac{1}{11!}[3^{11}\alpha_0+2^{11}\alpha_1 + \alpha_2 + \alpha_4 + 2^{11}\alpha_5 + 3^{11}\alpha_6 + 4^{11}\alpha_7 + 5^{11}\alpha_8]$$
$$- \frac{1}{10!}[-3^{10}\beta_0 - 2^{10}\beta_1 - \beta_2 + \beta_4 + 2^{10}\beta_5 + 3^{10}\beta_6 + 4^{10}\beta_7 + 5^{10}\beta_8] = 0$$

$$D_{12} = \frac{1}{12!}[3^{12}\alpha_0+2^{12}\alpha_1 + \alpha_2 + \alpha_4 + 2^{12}\alpha_5 + 3^{12}\alpha_6 + 4^{12}\alpha_7 + 5^{12}\alpha_8]$$
$$- \frac{1}{11!}[-3^{11}\beta_0 - 2^{11}\beta_1 - \beta_2 + \beta_4 + 2^{11}\beta_5 + 3^{11}\beta_6 + 4^{11}\beta_7 + 5^{11}\beta_8] = 0$$

However, on inserting the values we have obtained for the α_j into these equations we have:

$$-3\beta_0 - 2\beta_1 - \beta_2 + 2\beta_5 + 3\beta_6 + 4\beta_7 + 5\beta_8 = [10 - 25(a+b) + 8ab]$$
$$3^2\beta_0 + 2^2\beta_1 + \beta_2 + 2^2\beta_5 + 3^2\beta_6 + 4^2\beta_7 + 5^2\beta_8 = \frac{1}{3}[160 - 272(a+b) + 32ab]$$
$$-3^3\beta_0 - 2^3\beta_1 - \beta_2 + 2^3\beta_5 + 3^3\beta_6 + 4^3\beta_7 + 5^3\beta_8 = \frac{1}{4}[560 - 880(a+b) + 64ab)]$$
$$3^4\beta_0 + 2^4\beta_1 + \beta_2 + 2^4\beta_5 + 3^4\beta_6 + 4^4\beta_7 + 5^4\beta_8 = \frac{1}{5}[3400 - 3656(a+b) + 168ab]$$
$$-3^5\beta_0 - 2^5\beta_1 - \beta_2 + 2^5\beta_5 + 3^5\beta_6 + 4^5\beta_7 + 5^5\beta_8 = \frac{1}{6}[14960 - 13552(a+b) + 256ab)] \quad (9)$$
$$3^6\beta_0 + 2^6\beta_1 + \beta_2 + 2^6\beta_5 + 3^6\beta_6 + 4^6\beta_7 + 5^6\beta_8 = \frac{1}{7}[80440 - 53912(a+b) + 512ab]$$
$$-3^7\beta_0 - 2^7\beta_1 - \beta_2 + 2^7\beta_5 + 3^7\beta_6 + 4^7\beta_7 + 5^7\beta_8 = \frac{1}{8}[384320 - 208960(a+b) + 1024ab]$$
$$3^8\beta_0 + 2^8\beta_1 + \beta_2 + 2^8\beta_5 + 3^8\beta_6 + 4^8\beta_7 + 5^8\beta_8 = \frac{1}{9}[1973320 - 827336(a+b) + 2048ab]$$
$$-3^9\beta_0 - 2^9\beta_1 - \beta_2 + 2^9\beta_5 + 3^9\beta_6 + 4^9\beta_7 + 5^9\beta_8 = \frac{1}{10}[9707601 - 3260752(a+b) + 4100ab]$$

We can satisfy these equations by arbitrarily choosing: $\beta_2 = \beta_6, \beta_1 = \beta_7, \beta_0 = \beta_8$. Then (9) becomes;

$$2\beta_0 + 2\beta_1 + 2\beta_2 + 2\beta_5 = [10 - 25(a+b) + 8ab]$$
$$34\beta_0 + 20\beta_1 + 10\beta_2 + 4\beta_5 = \frac{1}{3}[160 - 272(a+b) + 32ab]$$
$$98\beta_0 + 56\beta_1 + 26\beta_2 + 8\beta_5 = \frac{1}{4}[560 - 880(a+b) + 64ab]$$
$$706\beta_0 + 210\beta_1 + 82\beta_2 + 16\beta_5 = \frac{1}{5}[3400 - 3656(a+b) + 168ab]$$
$$2882\beta_0 + 992\beta_1 + 242\beta_2 + 32\beta_5 = \frac{1}{6}[14960 - 13552(a+b) + 256ab] \quad (10)$$
$$16354\beta_0 + 4160\beta_1 + 730\beta_2 + 64\beta_5 = \frac{1}{7}[80440 - 53912(a+b) + 512ab]$$
$$75938\beta_0 + 16256\beta_1 + 2186\beta_2 + 128\beta_5 = \frac{1}{8}[384320 - 208960(a+b) + 1024ab]$$
$$397186\beta_0 + 65792\beta_1 + 6562\beta_2 + 256\beta_5 = \frac{1}{9}[1973320 - 827336(a+b) + 2048ab]$$
$$1933442\beta_0 + 261632\beta_1 + 19682\beta_2 + 512\beta_5 = \frac{1}{10}[9707601 - 3260752(a+b) + 4100ab]$$

Then solving the equations for values of $\beta_0, \beta_1, \beta_2$ and β_5 give:

$$2\beta_0 + 2\beta_1 + 2\beta_2 + 2\beta_5 = [10 - 25(a+b) + 8ab]$$

Making β_5 the subject of the formulae

$$\beta_5 = \frac{1}{2}[10 - 25(a+b) + 8ab) - \beta_0 - \beta_1 - \beta_2] \qquad (11)$$

Put (11) into

$$34\beta_0 + 20\beta_1 + 10\beta_2 + 4\left[\frac{1}{2}[10 - 25(a+b) + 8ab) - \beta_0 - \beta_1 - \beta_2]\right] = \frac{1}{3}[160 - 272(a+b) + 32ab]$$

$$\beta_2 = \frac{5}{8}(150 - 247(a+b) + 24ab) + 30\beta_0 - 16\beta_1 \qquad (12)$$

Putting (12) into (11)

$$\beta_5 = \frac{2}{9}[-140 + 22(a+b) - 16ab] - 31\beta_0 + 15\beta_1 \qquad (13)$$

Putting (12) and (13) into the following equations

$$98\beta_0 + 56\beta_1 + 26\beta_2 + 8\beta_5 = \frac{1}{4}[560 - 880(a+b) + 64ab]$$

$$706\beta_0 + 210\beta_1 + 82\beta_2 + 16\beta_5 = \frac{1}{5}[3400 - 3656(a+b) + 168ab]$$

respectively;

Therefore we have;

$$630\beta_0 - 296\beta_1 = \frac{640}{36}(520 - 655(a+b) + 56ab) \qquad (14)$$

and

$$706\beta_0 + 210\beta_1 = -\frac{21148}{810}(3360 - 3431(a+b) + 169ab) \qquad (15)$$

Making β_1 the subject in (15)

$$\beta_1 = -\frac{21148}{170100}(3360 - 3431(a+b) + 169ab) - \frac{706}{201}\beta_0 \qquad (16)$$

Put (16) into (14)

$$\beta_0 = -\frac{5}{100{,}000}(-994040 + 1014921(a+b) - 473404ab) = \beta_8 \qquad (17)$$

Put (17) into (16)

$$\beta_1 = -\frac{3}{25}(102764 - 1018352(a+b) + 47464ab) = \beta_7 \qquad (18)$$

Putting (17) and (18) into (12) and (13) to obtain values for β_2 and β_5;

$$\beta_2 = \frac{25}{10}(891726 - 2033520(a+b) - 425916ab) = \beta_6 \qquad (19)$$

and

$$\beta_5 = \frac{1}{50}(-1096944 + 2033295(a+b) - 52084ab) \qquad (20)$$

β_4 vanishes but of $\beta_3 \neq 0$ since $r = 3$

Were D_9 and D_{10} are the error constants which we will not necessarily solve for the purpose of eliminating unnecessarily ambiguity:

Therefore, since a = cosθ1, b = cosθ2, 0 < θ1 < π, 0 < θ2 < π, a and b are restricted to the range -1 < a < 1 and -1 < b < 1. Our choice of values for a and b is guided by the fact that we like to minimize the error constant as well as the need to develop a method that makes computation easier by reducing the number of operations involved. The following values are therefore, assigned to the variables: a = 3/4, b = -1/3.

Hence the following values are obtained for the coefficients α_i, β_i:

$$\alpha_8 = +1, \alpha_7 = -\frac{5}{4}, \alpha_6 = -\frac{5}{6}, \alpha_5 = 0, \alpha_4 = 0, \alpha_3 = -0, \alpha_2 = \frac{5}{6}, \alpha_1 = \frac{5}{4}, \alpha_0 = -1$$

$$\beta_2 = -\frac{185810708}{10} = \beta_6, \beta_1 = -\frac{1000246}{25} = \beta_7, \beta_0 = -\frac{226}{10} = \beta_8, \beta_5 = -\frac{119517}{50}$$

Substituting the values of a and b into (20), the Error constant is: -0.002489711924.

And finally, we have the scheme:

$$\alpha_8 y_{n+8} + \alpha_7 y_{n+7} + \alpha_6 y_{n+6} + \alpha_2 \frac{5}{6} y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n$$
$$= h[\beta_8 f_{n+8} + \beta_7 f_{n+7} + \beta_6 f_{n+6} + \beta_5 f_{n+5} + \beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n]$$

Therefore substituting values of α_i and β_i we have the scheme of order 10 degree 8 (8-step sizes) to be;

$$y_{n+8} - \frac{5}{4} y_{n+7} - \frac{5}{6} y_{n+6} + \frac{5}{6} y_{n+2} + \frac{5}{4} y_{n+1} - y_n$$
$$= h\left[\left(-\frac{226}{10}\right) f_{n+8} + \left(-\frac{1000246}{25}\right) f_{n+7} + \left(-\frac{185810708}{10}\right) f_{n+6} + \left(-\frac{119517}{50}\right) f_{n+5} + \left(-\frac{185810708}{10}\right) f_{n+2} + \left(-\frac{1000246}{25}\right) f_{n+1} + \left(-\frac{226}{10}\right) f_n\right]$$

Numerical Examples

Using the scheme to solve some differential equations as shown in Ndanusa & Adebayo (2008).

Here, as with all the k-steps methods (k<1) we need to use another method to calculate additional starting values. The Fourth Order Runge-kutta method is used to evaluate the starting values y_n, n= 0,1,2,3,…5, since the Runge-Kutta methods constitute the most efficient method for generating starting values for linear multistep methods. The Fourth Order Runge-Kutta method is given below:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n),\ k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right),\ k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right),\ k_4 = f(x_n + h, y_n + hk_1)$$

We choose as our predictor, fourth order Adams-Bashford method:

$$y_{n+10} = y_{n+9} + h\left[\left(-\frac{226}{10}\right) f_{n+8} + \left(-\frac{1000246}{25}\right) f_{n+7} + \left(-\frac{185810708}{10}\right) f_{n+6} + \left(-\frac{119517}{50}\right) f_{n+5} + \left(-\frac{185810708}{10}\right) f_{n+2} + \left(-\frac{1000246}{25}\right) f_{n+1} + \left(-\frac{226}{10}\right) f_n\right]$$

Table1. Problem: F = x+y; Y(0) = 1; h = 0.1; exact Solution: Y(x) = $2e^{-x-1}$

	Exact	Y(x)	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.1103418362	1.1103416667	1.6948462878E-07
0.2	1.2428055163	1.2428051417	3.7461895075E-07
0.3	1.3997176152	1.3997169941	6.2102693121E-07
0.4	1.5836493953	1.5836484802	9.1512116951E-07
0.5	1.7974425414	1.7974412772	1.2642065803E-06
0.6	2.0442376008	2.0442361876	1.4132161703E-06
0.7	2.3275054149	2.3275025204	2.8945628872E-06
0.8	2.6510818570	2.6510783589	3.4980860688E-06
0.9	3.0192062223	3.0192004614	5.7608878645E-06
1.0	3.4365636569	3.4365566462	7.0107547572E-06

Table 2. Problem: F = $3x^2$- $6x+5$; Y(0) =1; $h = 0.1$; Exact Solution: Y(x) = $x^{3-}3x^2+5x+1$

	Exact	Y(x)	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.4710000000	1.4710000000	0.0000000000E+00
0.2	1.8880000000	1.8880000000	0.0000000000E+00
0.3	2.2570000000	2.2570000000	0.0000000000E+00
0.4	2.5840000000	2.5840000000	0.0000000000E+00
0.5	2.8750000000	2.8750000000	0.0000000000E+00
0.6	3.1360000000	3.1360000000	0.0000000000E+00
0.7	3.3730000000	3.3730000000	0.0000000000E+00
0.8	3.5920000000	3.5920000000	0.0000000000E+00
0.9	3.7990000000	3.7990000000	0.0000000000E+00
1.0	4.0000000000	4.0000000000	0.0000000000E+00

II. Discussion of Results

In Table 1, the scheme produced errors as shown, within acceptable limits. The scheme is very accurate as exhibited in Table 2; there is no error at all. This is understandable because the solution of the differential equation is a polynomial of degree three k that is not greater than six, the step number.

III. Conclusion

From previous numerous solutions of differential equations deduced by numerical method it is safe to conclude that the method is accurate as it produced results which are comparable with those produced by other similar methods.

IV. REFERENCES

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