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Area Enclosed by Tangential Circles

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ABSTRACT

The problem in this paper finds its roots in trigonometry and recreational pure mathematics. Given n tangential circles having radius r, the problem asks to find the area enclosed by these circles. The paper takes a logical, step-by-step approach to break down the problem into several sub-parts, and then use those to derive the final solution.

Keywords : Circles, Trigonometry, Trigonometric Calculations

I. INTRODUCTION

Question 32 of the Grade 10 CBSE Board Exams 2011 asked students to find the area enclosed by 3 identical circles mutually tangential to each other, as shown by *Figure 1*. While solving the question involved some basic algebraic and trigonometric calculations, it is interesting to explore how the idea and logic behind this question can be expanded. This paper aims to find a general expression that can be used to calculate the area enclosed by n tangential circles having a radius r.

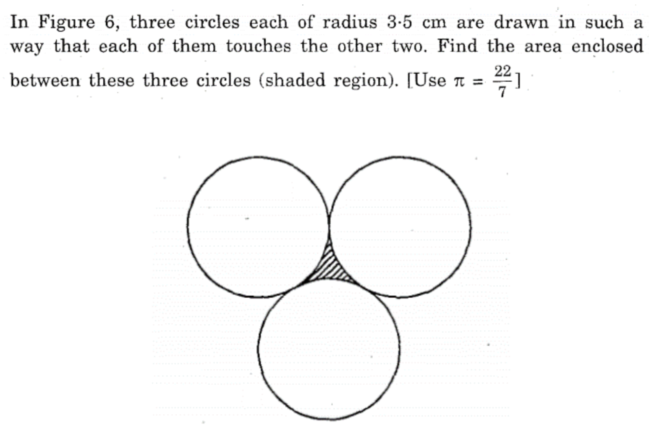

Figure 1: Q32, Y10 CBSE 2011

II. APPROACHING THE PROBLEM

In order to understand how a general expression could be derived, this section of the paper will outline the approach that would be taken. This will be done by considering a base case, where n, or the number of circles, equals 3, as shown in *Figure 1*. The radius, r, of the circles will also be considered as 1 for this section.

The first step to the solution involves forming an equilateral triangle of side length $2r$, the edges of which would lie on the centres of each circle, forming an image similar to that in *Figure 2*.

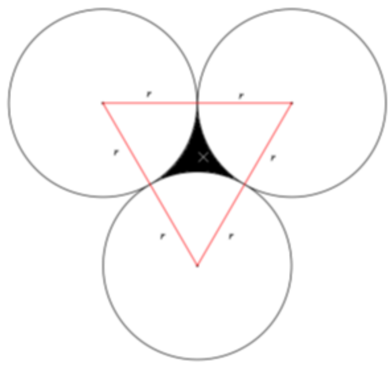

Figure 2: Visually understanding the approach

Next, the area of the triangle would be calculated using the formula:

$$A = \frac{1}{2} bc \sin A$$

In this case, it would yield:

$$\frac{1}{2} \cdot 2r \cdot 2r \cdot \sin 60$$

$$= \frac{1}{\not 2} \cdot \not 2 r \cdot \not 2 r \cdot \frac{\sqrt{3}}{\not 2} = r^2\sqrt{3}$$

Since the enclosed area needs to be calculated, the deference between the area of the triangle and the area of each circle's sector that form with the same interior angles as the triangle can be calculated, as shown below in Figure 3.

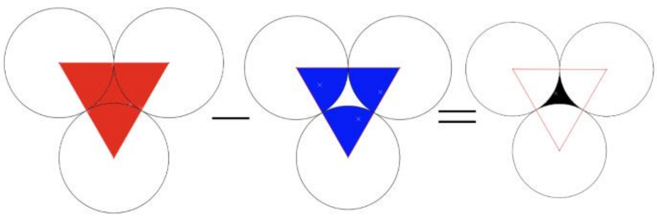

Figure 3: Visually understanding the approach

The area of each sector can be summed and then subtracted from the area of the triangle to return the enclosed area:

$$A_{\text{triangle}} - 3 \cdot A_{\text{sector}} = A_{\text{enclosed}}$$

$$r^2\sqrt{3} - 3 \cdot \left(\pi r^2 \cdot \frac{\pi/3}{2\pi} \right) = A_{\text{enclosed}}$$

$$\therefore A_{\text{enclosed}} = r^2 \left(\sqrt{3} - \frac{\pi}{2} \right)$$

Since the radius is to be taken as $r = 1$, the area enclosed by 3 circles is $\sqrt{3} - \frac{\pi}{2}$ units, or ≈ 0.1612 units.

III. DERIVING THE SOLUTION

In this section of the paper, the logic explained in the earlier section is going to be applied in an orderly fashion in order to derive the solution to the problem.

Step 1: In the earlier section, an equilateral triangle was formed using the centre of each circle. However, since the three will now be n circles, it is safe to say that a regular n-sided polygon must be formed, the edges of which must lie on the centre of each circle. The side length of this regular polygon will be 2r, as shown earlier.

Step 2: The next step is to calculate the area of this polygon. Since every regular n-sided polygon can be split into n isosceles triangles, the area of the polygon can simply be calculated by multiplying the area of one such triangle by n. Figure 4 shows how splitting the isosceles triangle into 2 right angled triangles can facilitate the calculations.

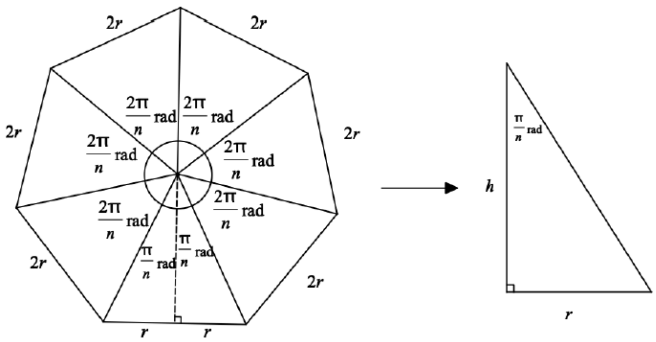

Figure 4 : Triangles in a regular polygon with n sides

Using basic trigonometry, the height of the right-angled triangle can be calculated, which can then be used to calculate the area of the isosceles triangle:

$$\tan\frac{\pi}{n} = \frac{r}{h}$$

$$\boxed{\therefore h = \frac{r}{\tan\frac{\pi}{n}}}$$

$$A_{isosceles} = \frac{1}{2}\cdot \text{base}\cdot h = \frac{1}{\cancel{2}}\cdot \cancel{2}r\cdot \frac{r}{\tan\frac{\pi}{n}}$$

$$= \frac{r^2}{\tan\frac{\pi}{n}}$$

As explained earlier, the area of one isosceles triangle must be multiplied by n to obtain the area of the polygon:

$$\boxed{A_{polygon} = \frac{nr^2}{\tan\frac{\pi}{n}}}$$

Step 3: The penultimate step is to calculate the area of the sectors of each circle, and then the last step would be to subtract that value from the area of the polygon, as demonstrated by Figure 3. To determine this, the interior angles of the polygon must be calculated, since the area of the sector is based on those. It is known that the interior angles of a regular polygon are given by: $\frac{\pi(n-2)}{n}$. This expression can be used to determine the area of any sector, the value of which can be multiplied by n to obtain the total area of all the sectors:

$$n\cdot \pi r^2 \cdot \frac{\frac{\cancel{\pi}(n-2)}{n}}{2\cancel{\pi}}$$

$$\cancel{n}\cdot \pi r^2 \cdot \frac{n-2}{2\cancel{n}}$$

$$\boxed{A_{sectors} = \frac{\pi r^2(n-2)}{2}}$$

Step 4: Finally, the last step is to simply subtract Asectors from Apolygon to obtain the remaining Aenclosed:

$$A_{enclosed} = A_{polygon} - A_{sectors}$$

$$= \frac{nr^2}{\tan\frac{\pi}{n}} - \frac{\pi r^2(n-2)}{2}$$

$$\boxed{A_{enclosed} = r^2\left(\frac{n}{\tan\frac{\pi}{n}} - \frac{\pi(n-2)}{2}\right), \forall n \in \mathbb{Z} \geq 3}$$

IV. CONCLUSION

This paper examined the relationship between the number of tangential circles and the area enclosed by them. The problem was initially approached through the lens of an actual exam question, and then the same logic was used to apply it to a general scenario with n circles with radius r. The approach involved calculating the areas of the regular polygon formed by connecting each circle's centre to the adjacent circle's centre, and of the sectors formed in each circle by the edges of said polygon. The derived formula was a difference of the two:

$$A_{enclosed} = \underbrace{\left(\frac{nr^2}{\tan{\pi/n}}\right)}_{A_{\text{Polygon}}} - \underbrace{\left(\frac{\pi r^2(n-2)}{2}\right)}_{A_{\text{Sectors}}}$$

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