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An Overview of Various Analytical Methods for Solving One Dimensional Wave Equation

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ABSTRACT

Partial Differential Equations has many physical applications in various fields such as Hydraulics, Mechanics and Theory of elasticity and so on. They have much wider range of application than Ordinary Differential equations which can model only the simplest physical system. Laplace equation, Navier-Stokes, Wave& Heat equations play a vital role in Fluid Dynamics & Electromagnetism. Schrodinger's equation constitutes a fundamental part of Quantum Physics. In Partial Differential Equations, one dimensional wave equation is one of the major mathematical problems whose governing equation signifies transverse vibrations of an elastic string. To get the solution of wave equation, various analytical as well as numerical methods are available..In the present article, we take an overview of some of the analytical methods. Separation of Variables is most commonly used method for wave equations. In which, given function is expressed as a product of two single variable functions which reduces the partial differential equation to two ordinary differential equations. Determining the solution of these ordinary differential equations with boundary conditions defines the general solution of wave equation. D'Alembert's method is transforming the partial differential equation by introducing two new independent variables corresponding to an explicit solution of wave equation along with boundary conditions. In Laplace transform method, the transform is used with respect to one of the variables. This converts to an ordinary differential equation, which gives the solution by boundary conditions. Another approach to find the solution using finite Fourier sine transform is also cited here.

Keywords : Wave equation, Laplace Transform, Partial Differential equation, Computational Mathematics.

I. INTRODUCTION

In the present paper, we take an overview of various analytical methods of solving one dimensional wave equation. One dimensional wave equation along with initial and boundary condition constitutes initial boundary value problem (IBVP), which occurs frequently in many physical phenomena. We obtain an analytical solution of IBVP by separation of variable method, D' Alembert's method , Laplace transform method and Finite Fourier sine transform[5]. The vibration of string problem was first solved by John Bernoulli, a Swiss Mathematician. He has assumed the string as a flexible thread which has a finite number of equally distanced beads or weights placed along it. This was time independent equation.

But Jean Le Rond D'Alembert, a French Mathematician, introduced time variable and derived one dimensional wave equation in 1746. In 1750, Swiss Mathematician, Leonard Euler gave a solution of the Wave equation using Fourier series. Pierre-Simon de Laplace has given the solution to potential equation in the study of Gravitational pull. Joseph Fourier has derived heat equation and Solution by separation of variables in 1807 [1]. The rest of this article is systematized as follows. Section 2 talks about mathematical formulation of the IBVP. Section 3 describes solution of Wave equation using method of separation of variables, D' Alembert's method , Laplace transform method and Finite Fourier sine transform method. Section 4 summarizes the article.

II. MATHEMATICAL FORMULATION OF THE PROBLEM:

Consider a uniformly stretched elastic string with length a, which is fixed at two points O & A. Constant tension τ is applied on it. The tension τ is to be considered large than the weight of the string so that the gravitational pull becomes negligible.

Let the string be released from rest and allowed to vibrate [1, 2].

The problem is to study the motion of the string with no external forces acting on it. Assume that each point of the string makes a small vibration at right angle to the equilibrium position of the string in one place.

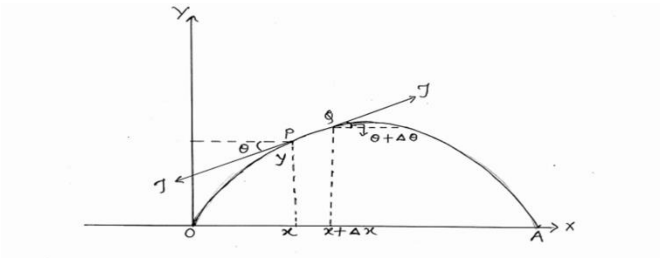

Figure No-1 vibrating string

Let the motion of the string takes place in XY plane. String is fixed at O & A on the X-axis. Let the string be in a position OPA at time t. Consider motion element P(x, y) & Q $(x + \Delta x, y + \Delta y)$ where tangents make angle $\theta \;\&\; \theta + \Delta\theta$ with X axis respectively. The element is moving upward with acceleration $\frac{\partial^2 y}{\partial t^2}$. The vertical component of force is acting on it is given as

$$\begin{aligned} F &= \tau \sin(\theta + \Delta\theta) - \tau \sin(\theta) \\ &= \tau(\sin(\theta + \Delta\theta) - \sin\theta) \\ &= \tau(\tan(\theta + \Delta\theta) - \tan\theta) \end{aligned}$$

$$= \tau \left[\left[\frac{\partial y}{\partial x}\right]_{x+\Delta x} - \left[\frac{\partial y}{\partial x}\right]_{x} \right],$$

where last equation is due to the fact that θ is very small

If μ is the mass of the string per unit length then by using Newton's 2nd law of motion

$$\mu \delta x \frac{\partial^2 y}{\partial t^2} = \tau \left(\left[\frac{\partial y}{\partial x}\right]_{x+\Delta x} - \left[\frac{\partial y}{\partial x}\right]_{x} \right),$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu} \left[\frac{\left[\frac{\partial y}{\partial x}\right]_{x+\Delta x} - \left[\frac{\partial y}{\partial x}\right]_{x}}{\Delta x} \right] \qquad (1)$$

Taking limits Q→ P in equation (1) we get $\Delta x \to 0$,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu} \frac{\partial}{\partial x} \left[\frac{\partial y}{\partial x}\right],$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu} \frac{\partial^2 y}{\partial x^2}.$$

Let $\frac{\tau}{\mu} = c^2 \Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is the classical one dimensional wave equation.

III. ANALYTICAL METHODS FOR WAVE EQUATIONS

3.1 Method of Separation of Variables[1,2]

Consider one dimensional wave equation

$\frac{\partial^2\eta}{\partial t^2} = c^2 \frac{\partial^2\eta}{\partial x^2}$ -------------- (I)

along with boundary conditions $\eta(0,t) = 0, \eta(a,t) = 0$

Let $\eta(x,t) = F(x)G(t)$, where F is a function of x & G is a function of t only. Then

$\frac{\partial\eta}{\partial t} = F\,\frac{dG}{dt}$ & $\frac{\partial\eta}{\partial x} = G\,\frac{dF}{dx}$,

$\frac{\partial^2\eta}{\partial t^2} = F\,\frac{d^2G}{dt^2}$ & $\frac{\partial^2\eta}{\partial x^2} = G\,\frac{d^2F}{dx^2}$.

And equation (I) becomes

$F\,\frac{d^2G}{dt^2} = c^2\,G\,\frac{d^2F}{dx^2}$.

Separating variables,

$\frac{\frac{d^2G}{dt^2}}{c^2G} = \frac{\frac{d^2F}{dx^2}}{F} = k$ or $\frac{G''}{c^2G} = \frac{F''}{F} = k$

$\frac{d^2G}{dt^2} - kc^2G = 0$ and

$\frac{d^2F}{dx^2} - kF = 0$, with boundary conditions $\eta(0,t) = 0,\ \eta(a,t) = 0\ \ \forall t$

However $\eta(x,t) = F(x)\,G(t)$, thus $\eta(0,t) = 0 \Rightarrow F(0)\ G(t) = 0$ &

$\eta(a,t) = 0 \Rightarrow F(a)\ G(t) = 0$

If $G(t) = 0 \Rightarrow k = 0 \Rightarrow Trivial\ solution$, hence $G(t) \neq 0$

$\therefore F(0) = 0$ & $F(a) = 0$

i. If $k = 0 \Rightarrow$

$$F'' = 0 \Rightarrow F(x) = C_1x + C_2$$

$$F(0) = 0 \Rightarrow C_2 = 0\ \&\ F(a) = 0 \Rightarrow C_1 = 0$$

$$\therefore F(x) = 0$$

We get trivial solution in this case.

ii. If $k > 0\ say\ k = m^2$

$\frac{F''}{F} = m^2 \Rightarrow D^2 - m^2 = 0$ is corresponding auxiliary equation

$F(x) = C_1e^{mx} + C_2e^{-mx}$

$F(0) = 0 \Rightarrow C_1 + C_2 = 0$ &

$F(a) = 0 \Rightarrow C_1e^{ma} + C_2e^{-ma}=0$

$\Rightarrow C_1 = 0\ \&C_2 = 0\ \therefore F(x) = 0$

We get trivial solution in this case.

iii. If $k < 0\ say\ k = -m^2, m > 0$

$\frac{F''}{F} = -m^2 \Rightarrow D^2 + m^2 = 0$ is corresponding auxiliary equation

$F(x) = C_1\cos mx + C_2\sin mx$,

$F(0) = 0 \Rightarrow C_1 = 0\ \&\ F(a) = 0 \Rightarrow$
$C_2\sin ma = 0 \Rightarrow \sin ma = 0$

where $ma = n\pi \Rightarrow m = {n\pi}/{a}$, $n = 1,2,3, ...$

Also, $\frac{G''}{c^2G} = -m^2 \Rightarrow$

$D^2 + c^2m^2 = 0 \Rightarrow$

$G(t) = C_3\cos cmt + C_4\sin cmt$

$G_n(t) = C_3\cos\left(\frac{n\pi ct}{a}\right) + C_4\sin\left(\frac{n\pi ct}{a}\right)$

The solution

$\eta(x,t) = \left(C_2\sin\left(\frac{n\pi x}{a}\right)\right)\left(C_3\cos\left(\frac{n\pi ct}{a}\right) + C_4\sin\left(\frac{n\pi ct}{a}\right)\right)$

Or $\eta(x,t) = \left(A_n\cos\left(\frac{n\pi ct}{a}\right) + B_n\sin\left(\frac{n\pi ct}{a}\right)\right)\left(\sin\left(\frac{n\pi x}{a}\right)\right)$

$\eta(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi ct}{a}\right) + B_n\sin\left(\frac{n\pi ct}{a}\right)\right)\left(\sin\left(\frac{n\pi x}{a}\right)\right)$

This is the general solution of one dimensional wave equation (1) along with boundary conditions.

3.2 D' Alembert's Method[1,2]

Consider $\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2}$ (1)

$where\ \ c^2 = {\tau}/{\mu}$

Let $\alpha = x + ct$ & $\beta = x - ct$ then $\eta = f(\alpha,\beta)$

$\therefore\ \eta_x = \eta_\alpha\alpha_x + \eta_\beta\beta_x \Rightarrow \eta_x = \eta_\alpha + \eta_\beta$ (2)

Similarly, $\eta_{xx} = \left(\eta_\alpha + \eta_\beta\right)_x$

$= (\eta_\alpha + \eta)_\alpha\alpha_x + \left(\eta_\alpha + \eta_\beta\right)_\beta\beta_x$

$\eta_{xx} = \eta_{\alpha\alpha} + 2\eta_{\alpha\beta} + \eta_{\beta\beta}$ (3)

Assuming all the partial derivatives are continuous, we get $\eta_{\alpha\beta} = \eta_{\beta\alpha}$

Now differentiating with respect to t, $\eta = \eta_\alpha\alpha_t + \eta_\beta\beta_t \Rightarrow \eta_t = c\eta_\alpha + (-c)\eta_\beta$

$\Rightarrow \eta_t = c(\eta_\alpha - \eta_\beta)$ (4)

$\&\eta_{tt} = c\left(\eta_\alpha - \eta_\beta\right)_t$

$= c\left(\eta_\alpha - \eta_\beta\right)_\alpha\alpha_t + c\left(\eta_\alpha - \eta_\beta\right)_\beta\beta_t$

$\eta_{tt} = c^2(\eta_{\alpha\alpha} - 2\eta_{\alpha\beta} + \eta_{\beta\beta})$ (5)

Substituting in equation (1)

$$c^2(\eta_{\alpha\alpha} - 2\eta_{\alpha\beta} + \eta_{\beta\beta}) = c^2(\eta_{\alpha\alpha} + 2\eta_{\alpha\beta} + \eta_{\beta\beta})$$

$\Rightarrow \eta_{\alpha\beta} = 0$ (6)

Integrating (6) with respect to β & then w. r. t $\alpha \Rightarrow$

$\frac{\partial\eta}{\partial\alpha} = f(\alpha)$

$$\& \ \eta = \int f(\alpha)d\alpha + g(\beta)$$

here $f(\alpha)$& $g(\beta)$ are arbitrary functions of α & β.

$\therefore \eta = \emptyset(\alpha) + g(\beta) \Rightarrow \eta(x,t) = \emptyset(x+ct) + g(x-ct)$ (7)

is the solution of wave equation by D' Alembert's method.

Now we determine $\emptyset$ & g with initial condition $\eta(x,0) = f(x)$

& $\frac{\partial\eta}{\partial t} = 0 \ at \ t = 0$.

Differentiating (7) with respect to t,

$\frac{\partial\eta}{\partial t} = c\,\emptyset'(x+ct) - c\,g'(x-ct)$,

Put $\frac{\partial\eta}{\partial t} = 0 \ at \ t = 0 \Rightarrow \emptyset'(x) = g'(x) \Rightarrow \emptyset(x) = g(x) + K$

Now put $\eta = f(x)$ at $t = 0$ in (7)

$$\Rightarrow \eta(x,0) = f(x) = \emptyset(x) + g(x)$$

$\Rightarrow f(x) = 2g(x) + K$,

$\Rightarrow g(x) = \frac{1}{2}(f(x) - K)$& $\emptyset(x) = \frac{1}{2}(f(x) + K)$.

From equation (7)

$$\eta(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

3.3 Laplace Transform Method[2]

Consider one dimensional wave equation

$\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2}$, $x > 0, t > 0$ (1)

With boundary conditions

$$\eta(0,t) = f(t),$$

$\eta(x,0) = 0, \lim_{x\to\infty} \eta(x,t) = 0$ & $\left(\frac{\partial\eta}{\partial t}\right)_{t=0} = 0.$

For semi-infinite string

For $t > 0$, string is in sine wave form [1,2]

Take Laplace transform of (1),

$\Rightarrow L[\eta_{tt}] = c^2L[\eta_{xx}]$,

$\Rightarrow s^2L[\eta] - s\eta(x,0) - \eta_t(x,0) = c^2L[\eta_{xx}]$,

$\Rightarrow s^2U(x,s) = c^2L[\eta_{xx}]$ (2)

Since $\eta(x,0) = 0$ & $\eta_t(x,0) = 0$,

Now $L[\eta_{xx}] = \int_0^\infty e^{-st}\frac{\partial^2\eta}{\partial x^2}\, dt$,

$= \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}\,\eta(x,t)\,dt$,

$= \frac{\partial^2}{\partial x^2}\,L[\eta(x,t)]$

$L[\eta_{xx}] = \frac{\partial^2}{\partial x^2}\,U(x,s)$ (3)

Put (3) in (2)

$$s^2U(x,s) = c^2\frac{\partial^2}{\partial x^2}\,U(x,s)$$

$\frac{\partial^2 U}{\partial x^2} - \frac{s^2}{c^2}\,U = 0$ (4)

Equation (4) is an ordinary differential equation for $U(x,s)$ and hence its solution is given as

$U(x,s) = P\,e^{sx/c} + Q\,e^{-sx/c}$ (5)

Where P & Q are arbitrary constants of s.

From boundary conditions,

$\eta(0,t) = f(t)$,

Let $L[f(t)] = F(s)$,

$\therefore U(0,s) = L[\eta(0,t)] = L[f(t)] = F(s)$.

Now

$\lim_{x\to\infty} U(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}\eta(x,t)dt$,

$\lim_{x\to\infty} U(x,s) = \int_0^\infty e^{-st}\lim_{x\to\infty}\eta(x,t)dt$,

$= 0$ given boundary condition.

And interchanging integration with limit

$P(s) = 0$ in (5) since $c > 0 \Longrightarrow$ for every positive s ,$e^{sx/c}$ increases as x increases

Assume $s > 0$

$\therefore U(0,s) = Q(s) = F(s)$

$\therefore$ (5) $\Longrightarrow U(x,s) = F(s)e^{-sx/c}$

Taking inverse Laplace transform, we obtain

$\eta(x,t) = f\left(t - \frac{x}{c}\right)\eta\left(t - \frac{x}{c}\right)$.

By using second shifting property $\{L^{-1}[e^{-as}F(s)] = f(t-a)\eta(t-a)\}$ we get the required solution

$$\Rightarrow \eta(x,t) = \begin{cases} \sin\left(t - \frac{x}{c}\right) & , \quad if\,\frac{x}{c} < t < \frac{x}{c} + 2\pi \\ 0 & , \text{otherwise} \end{cases}$$

3.4 Using Finite Fourier Sine Transform[6]

Consider one dimensional wave equation

$\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2}$, along with boundary condition

$\eta(0,t) = 0, \eta(a,t) = 0$

$F_s\left[\frac{\partial^2\eta}{\partial t^2}\right] = \int_0^a \frac{\partial^2\eta}{\partial x^2} sin\left(\frac{n\pi x}{a}\right) dx \ , n \in Z$

$\Rightarrow F_s\left[\frac{\partial^2\eta}{\partial t^2}\right] = F_s\left[c^2\frac{\partial^2\eta}{\partial x^2}\right]$

$\Rightarrow \int_0^a \frac{\partial^2\eta}{\partial t^2} sin\left(\frac{n\pi x}{a}\right) dx = c^2 \int_0^a \frac{\partial^2\eta}{\partial x^2} sin\left(\frac{n\pi x}{a}\right) dx$

$\Rightarrow \frac{\partial^2}{\partial t^2}\int_0^a \eta(x,t) sin\left(\frac{n\pi x}{a}\right) dx = c^2\left\{\left(\frac{\partial\eta}{\partial x} sin\left(\frac{n\pi x}{a}\right)\right)_0^a - \int_0^a \frac{\partial\eta}{\partial x} cos\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a} dx\right\}$

$\Rightarrow \frac{\partial^2}{\partial t^2}[F_s\,\eta(x,t)] = -\frac{c^2 n\pi}{a}\left[\left(\eta(x,t) cos\left(\frac{n\pi x}{a}\right)\right)_0^a - \int_0^a \eta(x,t)\, sin\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a} dx\right]$

$\Rightarrow \frac{\partial^2}{\partial t^2}[F_s\,\eta(x,t)] = -\frac{c^2 n\pi}{a}\left[(\eta(a,t)\cos n\pi - \eta(0,t)) - \int_0^a \eta(x,t)\, sin\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a} dx\right]$

$\Rightarrow \frac{\partial^2}{\partial t^2}[F_s\,\eta(x,t)] + \frac{c^2 n^2\pi^2}{a^2} F_s\,\eta(x,t) = 0$

This is second order ordinary differential equation.

Let $F_s\,[\eta(x,t)] = e^{mt}$ be trivial solution of the equation then $m = \pm\frac{in\pi c}{a}$

So the general solution is

$F_s[\eta(x,t)] = A\cos\left(\frac{n\pi ct}{a}\right) + B\sin\left(\frac{n\pi ct}{a}\right)$

Taking inverse Finite Fourier sine transform,

$\eta(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\left(A\cos\left(\frac{n\pi ct}{a}\right) + B\sin\left(\frac{n\pi ct}{a}\right)\right)\sin\left(\frac{n\pi x}{a}\right)$

IV. SUMMARY

Partial Differential equations have great importance in our daily life. They are used in Meteorology, Oceanography, Solar System, Economics, Physics, Chemistry & various Engineering branches to represent the model of Physical problem. We have taken an overview of some of the analytical methods used for solving one dimensional wave equation with boundary conditions. The general solution is obtained in each case. Our article takes a bird's eye view of various analytical methods of solving one dimensional wave equation where this equation is used in elastic mediums for modeling air column of an organ pipe and vibrations of a metal rod . These solutions will be helpful to all researchers working in the domain of Engineering and Science.

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