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Foundational View of Set Theory in Its Axiomatic Development

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ABSTRACT

Beingapowerful part and having its influence into all mathematical subjects, the set theory is having great significance. This paper reviews its arise in the form of axiomatic theory and how these axioms are used to develop the set theory. Subsets, cardinality of sets, its properties, basic set theory are used to proveseveral important results in real numbers.Because of continuing development in the theory of sets,this branch of mathematics shaped many basic concepts. This paper will reconnoitre problems where axiomatic set theory has salient role in its solution.

Keywords: Zermelo–Fraenkel axioms, ZFC, Countable sets.

I. INTRODUCTION

Set theory is a branch of mathematics used to systematize the data. Cantor's Continuum Hypothesis (CH) in 1878 clears that every infinite subset of **R** is either countable or have the same cardinality as that of **R** [1] is the main approach towards the development of set theory.He later worked on transfinite cardinal, ordinal numbers and the properties of real numbers especially topological properties of **R**. In the try of solution to many paradoxes like Russell's paradox, Burali-Forti paradox, Cantor's paradox, etc some axiomatic theories were proposed and studied.In the development of Set theory, Zermelo–Fraenkel axiomatic set theory in addition with axioms of choice ZFC and some additional natural axioms settled the continuum problem. It cannot be ignored that the meeting in Switzerland (1872) between Cantor and Dedekind is considered as the birth of this enormously successful subject called the set theory [2]. In this meeting the concept of belongingness to something having common properties radices the axioms. We must know belonging before going to axioms. If a belongs to A means a is an element contained in set A, it is denoted using mathematical symbol "$\in$" i.e. $a \in A$. Opposite to it, non-belonging is denoted by the symbol $\notin$. In the same period the study of function spaces started by Hilbert, Maurice Fréchet, and many others. Later Bernhard Riemann using sets and advanced study in set proposed ideas of topology, manifolds.

Ackermann [3] also proposed new and different axiomatic theory of sets and classes. Levy absorbedstandard ZF axiomatization and expanded it. The conceptsubset Dedekind defined special subsets following certain properties of Rings asIdeals and Ideals is the new branch of mathematics with lot of applications.

II. THEORETICAL AXIOMS

Some Theoretical Axioms are discussed below

Axiom 1.

Axiom of Extension is $\forall x\ (x \in A \text{ iff } x \in B) \rightarrow A = B$.

Any two sets are same if and only if they contain exactly same elements. Axiom of extension is the key in marking out the belonging.

Axiom 2.

For every set A and $x \in A$ with condition f(x), there exist a set B whose elements are same as that of set A satisfying the condition f(x). $B = \{x \in A \mid f(x)\}$.

This axiom using mathematical symbols can be expressed as $\forall A, \exists B\ \forall x(x \in B \leftrightarrow x \in A \wedge f(x))$. It raises the logical way towards another important subject called theory of subsets.

Axiom 3.

There exists a set that contains no elements and this set by axiom 1 is unique. Notation for the empty set is $\emptyset$ or $\{\}$.

$\exists A$ such that $A=\emptyset \wedge \forall x\ (x \notin A)$.

Empty set is a subset of every set, $\emptyset \subset A$.

Axiom 4.

Axiom of pairing states that for every xand every ythere exists a set C such that x and y are members of C.

$\forall x\ \forall y\ \exists C\ \forall z\ (z \in C \leftrightarrow z = x \vee z = y)$.

Axiom 5.

For every set A there exist a set C such that for all x in B where B is in A then x is also in C.

$\forall A \exists B\ \forall x(x \in B \leftrightarrow \exists C(x \in C \wedge C \in A)$

Axiom 6.

For every set A there is a set B consisting of the subsets of A, called as power set and denoted by P(A).

$\forall A \exists B \forall x[x \in B \leftrightarrow \exists C(C \subset A \wedge x=C)]$.

It can be interesting to see that the power set of empty set contains singleton element, the empty set itself.

For next axiom it is important to know the following definitions.

Definition 1.

(Injective) A function $f: A \rightarrow B$ is one-to-one if and only if for all $a, b \in A$, $f(a) = f(b)$ implies $a = b$.

Definition 2.

(Surjective) A function $f: A \rightarrow B$ is onto if for any $b \in B$ there exists an $a \in A$ for which $f(a) = b$.

Definition 3.

A function f is bijective if and only if it is both injective and surjective.

Definition 4.

The cardinality of a finite set A is the number of elements contained in a set, it is denoted as |A|.

Definition 5.

The symmetric difference oftwo sets A and B is the set of elements that are in oneand only one of the sets. The symmetric difference iswritten as A Δ B.

$A \Delta B = \{(A - B) \cup (B - A)\}$.

Definition 6.

The ordered pair of a and b is defined as $(a, b) = \{\{a\}, \{a, b\}\}$, denoted by the set (a, b).

Definition 7.

The Cartesian product of the family of sets {Ai} ($i \in I$), is the set of all families {ai} with ai $\in$ Ai for each $i \in I$.

The Cartesian Product $A \times B$ of two sets A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example: If $A = \{a, b\}$ and $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$.

Axiom 7.

Axiom of Choice:The Cartesian Product of a nonempty family of nonempty sets is nonempty.

Considera set of nonempty sets as Aand its union is theset P, then there existselement x contained in Xin A.

Zermelo–Fraenkel axioms together with axiom of choice isreferred as ZFC.

In the basic set theory Venn diagrams are generally used as it is the easiest way to solve problems or to prove some simple properties and easy to visualize. But Venn diagrams have some restrictions if it is concerned with the members of set, union of all subsets of a set cannot be easily represented using these Venn diagrams. Even it is not possible to use these diagrams if the number of sets are more and with different properties. Set theory is not having just a single method to solve related problems. Schoenfeld in [4] also said, "The phenomena we wish to "see" should affect our choice of method, and the choice of method will, in turn, affect what we are capable of seeing. And, of course, the kinds of claims one will be able to make (convincingly) will depend very much on the methods that have been employed"

With addition to axioms, it is essential to know some theorems [5] in set theory required to study sets in detail like subset of countable set is countable, if a set contains uncountable subset, then that set is also an uncountable, the intersection of finitely many countable sets is countable, The Cartesian product of finitely many countable sets is countable, there does not exist any surjection function from a set A to P(A), etc. Descriptive set theory (DST) is the specific study of definable sets of real numbers.Advanced to it completeness axiom applied in Bolzano's principle to observe the sequence of nested closed intervals in R.

III. EXAMPLES

Some examples where the definitions and axioms are usedto prove the results:

Example 1.Every finite set is countable

Example 2. The set of all integers Z is countable

Example 3. The set of all rational numbers is countable

Example 4.The set of real numbers in [0; 1] is uncountable.

Example 5. If A is countable and non empty set $B \subset A$, then B is also countable

Proof:

Case 1: If A is finite, being a subset B is also finite.

Case 2: If A is countably infinite, then there exist a bijective function $f : A \rightarrow N$. $f(B) \subset N$, since any subset of N is countable so f(B) is either finite or countably infinite. As B is equivalent to f(B) or B and f(B) are of same cardinality (since f is injective), hence B is countable.

Example 6. If $B \subset A$ and B is uncountable, then A is uncountable.

Proof:

This can be shown by contradiction.

Consider A is countable and then there will be two cases.

Case 1: If A is finite, then $B \subset A$, is a contradiction since B is uncountable.

Case 2: If A is infinitely countable, then there exist a bijective function $f : A \rightarrow N$, as $B \subset A$ implies that $f : B \rightarrow f(B)$ is also a bijective function. But $f(B) \subset f(A) = N \Rightarrow f(B) \subset N$, therefore f(B) is countable. Since there is a bijection from B to f(B) then cardinality of B and f(B) is same. This is also a contradiction since an uncountable set can never be equivalent to a countable set.

Example 7.

IfB is countable andf $:A\rightarrow B$ is injective; then A is countable.

Proof:

If Ais Finite, then it holds that A is countable. Now let Abe infinite. Since f is injective,Ais equivalent to f(A). Therefore f(A) is also infinite. As $f(A) \subseteq B$, hence Bis infinite. But given thatBis countable so Bis countably infinite. Subset of a countable set is countable hencef(A) iscountable. Ais equivalent tof(A) soAis also countable.

Example 8.

If A is countable and the function $f: A\rightarrow B$ is surjective then B is countable

Proof:

Given that f is surjective. Therefore, f has right-inverse $g: B\rightarrow A$, i.e. $f \circ g(b) = b, \forall b \in B$. The function g is injective since it has a left - inverse f.and if B is countable and $f: A \rightarrow B$ is injective; then A is countable. Since A is countable,B is countable (f is surjective).

IV. CONCLUSION

Axiomatic set theory is a very basic tool to have an idea about set theory and its development. Many applications of the set theory can also be seen in decision making problems, in data mining, soft computing, in the study of large cardinalconsistency.

In the case of inadequacy of parameters or where the data is not precise then one can use the advanced set theory concepts collaboratively even if the data is too large and is not fixed. Sets can be linearly ordered so limitations to axiom of choice can overcome by ordering principle.

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